On Norms of Derivations Implemented by Self-Adjoint Operators

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***ABSTRACT*—In the present paper, we introduce and study the concept of norms of derivations, in particular norm estimates of derivations implemented by self-adjoint operators. We show that $\|\delta_C\| = \|CX - XC\| \leq 2\|C\|$, for inner derivation while for generalized derivation we establish that $\|\delta_{C,D}\| = \|C\| + \|D\|$, for all $C, D, X \in B(H)$. We also estimate that $\|C\| \leq \|CX - XC\| \leq 2\|C\|$ and $\|\delta_C\| \geq 2(\|C\|^2 + \beta^2)^{\frac{1}{2}}$**

***Index Terms*—Generalized Derivation, Norms of derivations, Self-adjoint Operators.**

1 INTRODUCTION

The study of notion of norms is of great interest to many researchers and mathematicians. Of special interest is the determination of norms of derivations implemented by self-adjoint operators. By letting $B(H)$ denote the algebra of all bounded linear operators on an infinite-dimensional complex Hilbert space, we define the inner derivation as $\delta_A(X) = AX - XA$ and generalized derivation as $\delta_{A,B}(X) = AX - XB$, for operators $A, B \in B(H)$. Norms of elementary operators has been dealt with by quite a number of researchers. For instance the authors in [10] proved that for standard operator algebras on Hilbert spaces $\|M_{C,D} + M_{D,C}\| \geq 2(\sqrt{2} - 1)\|C\|\|D\|$. The research in [4] dealt with norm of a C^*-algebra and established that $\|CYD - DYC\| = 2\|C\|\|D\|$. The study in [14] investigated norms of elementary operators and in [15] focussed on computing the norm of elementary operators where it is shown that, $\|M_{C,D} + M_{D,C}\| \geq \|C\|\|D\|$. In [9], injective norm is used to characterize nomaloid operators and determine their lower norm estimates as, $\|C\|\|D\| \leq \|CYD + DYC\| \leq 2\|C\|\|D\|$. The authors in [6] determined the norm of an elementary operator and characterized these norms when they are implemented by norm-attainable operators. In their study they showed that $\|\mathcal{J}_{N,C,D}|B(H)\| \geq \|C\|\|D\|$, in which $C, D \in B(H)$ and $\mathcal{J}_{N,C,D}$ is a norm-attainable Jordan elementary operator. Others who studied this topic include [1,3,7,8,16].

2 DERIVATIONS

The concept of derivations came as a result of an elementary operators. An elementary operator is the sum basic elementary operator, i.e, $\mathcal{J} = \sum_{i=1}^{k} M_{C_iD_i}$ for all $C, D \in B(H)$. In this case derivations are examples of elementary operators. Therefore an inner derivation is defined as $\delta_A(X) = AX - XA$ while a generalized derivation as $\delta_{A,B}(X) = AX - XB$, for operators, $A, B, X \in B(H)$. Regarding the study of the norm of a derivation a number of results have been given. For instance, Stampfli and Williams [14] determined the upper norm estimate of inner derivation as $\|\delta_A\| \leq 2\|A\|$ and $\|\delta_{A,B}\| \leq \|A\| + \|B\|$, for generalized derivation. Stampfli [12] showed that the lower norm estimates of an inner derivation lie between two values i.e, $\|A\| \leq \|\delta_A(X)\| \leq 2\|A\|$.
Stampfli [12] in his study of self-adjoint derivation ranges established that the results of the propositions below:

Proposition 1. *$W_0(A)$ is a non-empty closed convex set included in $W(A)$.*

Proposition 2. *Let $A \in B(H)$, where H is a complex Hilbert space of* $\dim \geq 2$. *Then,*
$0 \in W_0(A) \Leftrightarrow \sup\{\|AX - XA\|, \|x\| = 1 = 2\|A\|\}$.

Proposition 3. *Let $A \in B(H)$, where H is a complex Hilbert space of* $\dim \geq 2$. *Then,*

(i) $0 \in W_0(A) \Rightarrow \|A\|^2 + |\lambda|^2 \leq \|A + \lambda I\|^2, \forall \lambda \in \mathbb{C}$
(ii) $\|A\| \leq \|A + \lambda I\| \Rightarrow 0 \in W_0(A)$
(iii) $\forall A \in B(H)$, *there exists a unique λ_0 such that* $\|A + \lambda_0 I\| \leq \|A - \lambda I\|, \forall \lambda \in \mathbb{C}$

Proposition 4. *Let $A, B \in B(H)$, such that $A \neq 0, B \neq 0$ with $\dim \geq 2$. Then the following condition are equivalent*

(i) $W_N(A) \bigcap W_N(-B) \neq \phi$,
(ii) There exists a sequence of operator s $\{X_n\}$ in $B(H)$ such that $\|X_n\| = 1, \lim_n \|AX_n - X_nB\| = \|A\| + \|B\|$,
(iii) $\|A\| + \|B\| \leq \|A + \lambda I\| + \|B + \lambda I\|$.

While studying derivation ranges, Stampfli [11] stated various propositions and corollaries;

Proposition 5. *Let $0 \leq A \leq I$ and $0 \leq B \leq I$. Then $ReAB \geq \frac{-1}{8}$. More generally,*
$ReAB \geq k_1k_2 - (K_1 - k_1)(K_2 - k_2)/8$,
$0 \leq k_1 \leq A \leq K_1 \leq B \leq K_2$.

Corollary 1. *Let T be a normal (hyponormal) operator. Then*

$$\|\delta_T\| \sup\{\|TA - AT\| : A \in B(H), \|A\| = 1\} = 2R_T,$$

where R_T is the radius of spectrum of T.

Corollary 2. *Let $0 \leq A \leq 1, 0 \leq B \leq 1$. Then $\|AB - BA\| = 2\|ImAB\| \leq \frac{1}{2}$.*

Okelo, Agure and Ambogo [6] determined the norm of an elementary operator using spectral decomposition concept in which they gave the following results.

Lemma 1. *If $J \in B(H)$, then there exists two isometries $\alpha_1, \alpha_2 \in B(H)$ such that $T = 1/2(\alpha_1 + \alpha_2)$. In addition, if $\dim N(J) = \dim N(J^*)$, then α_1, α_2 can be taken as unitaries.*

Nyamwala and Agure [5] determined the norm of elementary operators induced by normal operators in a finite-dimensional space using spectral resolution theorem.

Lemma 2. *Let B be a normal operator such that $B : H \to H$, where H is a finite n-dimensional space, then $\|B\| = \left(\sum_{i=1}^{n} |\alpha_i|^2\right)^{1/2}$ and α_i are distinct eigenvalues of B for corresponding eigenspaces $N_i, i = 1, 2, ...n$.*

3 NORMS OF DERIVATIONS

In this section we give the results of our study. We establish norms of derivations in the following ways:

Proposition 6. *Let $C \in B(H)$ where H is a complex Hilbert space and let λ_0 be the center of C.*

(i) $\|\delta_C\| = 2\|C - \lambda_0\| = 2inf\|C - \lambda\|, \lambda \in \mathbb{C}$
(ii) if $\beta \in W_0(C)$, then $\|\delta_C\| \geq 2(\|C\|^2 - \beta^2)^{\frac{1}{2}}$.

Proof.. (i) If $\dim H = 1$ the proof is evident. Suppose $\dim H \geq 1$. We establish that $0 \in W_0(C) \iff 0$ is the center of $C : \|C\| \leq \|C + \lambda\|$, for all $\lambda \in \mathbb{C}$. Which is equivalent to $\sup \|CY - YC\|, \|y\| = 1 = 2\|C\|$. Since $\delta_C = \delta_{C-\lambda I}$, the second equivalence fix the value of $\|\delta_C\|$ with the choice of λ imposed by the first equivalence.

(ii) For $\beta \in W_0(C)$ we associate a sequence $\{y_k\}$ with $\|y_k\| = 1, \lim_k \|Cy_k\| = \|C\|, \beta = \lim_k(Cy_k, y_k)$ and $G_k = Vect\{y_k, y'_k\}$, where y_k, y'_k is an orthonormal basis of G_k and $(Cy_k, y'_k) \geq 0$, where $Cy_k \in G_k$.
Let $Y_k = y_k \otimes y_k - y'_k \otimes \ y'_k$. Then

$$\begin{aligned}(\delta_C Y_k)y_k &= Cy_k - (Cy_k, y_k)y_k + (Cy_k, y'_k)y'_k \\ &= 2(Cy_k, y'_k)y'_k \\ &= 2(\|Cy_k\|^2 - (|(Cy_k, y_k)|^2)^{\frac{1}{2}} y'_k.\end{aligned}$$

Hence $\|\delta_C\| \geq \lim_k \|(\delta_C Y_k)y_k\| \geq 2(\|C\|^2 - |\beta|^2)^{\frac{1}{2}}$. □

Proposition 7. *Let C, D be two elements of $B(E)$, where E is a complex Hilbert space. Then*

(i) $\|\delta_{C,D}\| = inf\|C - \lambda\| + \|D - \lambda\|, \lambda \in \mathbb{C}$,
(ii) $W_N(C) \bigcup W_N(D) \neq \Phi \iff \|\delta_{C,D}\| = \|C\| + \|D\|$.

Proof.. In the study of $W_N(A)$ it was established that $\|C\| + \|D\| \leq \|C - \lambda\| + \|D - \lambda\| \iff \exists \{Y_k\}, \|Y_k\| = 1$, such that $\lim_k \|CY_k - Y_kD\| = \|C\| + \|D\|$. Since $\delta_{C,D}(Y) = \delta_{C-\lambda,D-\lambda}$ hence $\|\delta_{C,D}(Y)\| \leq \|C - \lambda\| + \|D - \lambda\|$, for all, $Y \in B(H), \|Y\| = 1$.
Then $\|\delta_{C,D}\| \leq inf\|C - \lambda\| + \|D - \lambda\|, \lambda \in \mathbb{C}$. □

Lemma 3. *Let $\alpha \in W_0(A)$. Then $\|\delta_A\| \geq 2(\|A\|^2 - |\alpha|^2)^{\frac{1}{2}}$*

Proof.. Note that $\|\delta_A\| = \sup\{\|AX - XA\| : X \in B(H)$ and $\|x\| = 1\}$. Since $\alpha \in W_0(A)$, there exists $u_n \in H$ such that $\|u_n\| = 1, \|Au_n\| \to \|A\|$ and $(Au_n, u_n) \to \alpha$. Set $Au_n = \mu u_n + \beta v_n$ where $(u_n, v_n) = 0$. Set $R_n u_n = u_n, R_n v_n = -v_n$ and $R_n = 0$ on $\{u_n, v_n\}$. Then $\|(AR_n - R_nA)u_n\| = 2|\beta_n| \geq 2(\|T\| - |b_n|^2)^{\frac{1}{2}} - \lambda_n$ where $\lambda \to 0$. Since $b_n \to \alpha$ hence the proof. □

Theorem 3. $\|\delta_A\| = 2\|A\|$ *if and only if* $0 \in W_0(A)$.

Proof.. It follows from the above lemma that $\|\delta_A\| \geq 2\|A\|$ if $0 \in W_0(A)$. Since $\|\delta_A\| \leq 2\|A\|$ sufficiency is proved. Suppose $\|\delta_A\| \leq 2\|A\|$ and so there exist u_n and X_n such that $\|u_n\| = \|X_n\| = 1$ and $\|AX_nu_n\| \to \|A\|$. Moreover, since$\|(AX_n - X_nA)u_n\| \to 2\|A\|, AX_nu_n = -X_nAu_n + \overline{\lambda}_n$ where $\|\overline{\lambda}\| \to 0$. Let $(Au_n, u_n) \to \alpha$ by choosing subsequence if necessary i.e $\alpha \in W_0(A)$. Observe that $(AX_nu_n, X_nu_n) = -(X_nA, X_n^*X_nu_n) = -(Au_n, u_n) + \lambda'_n$. Thus $\lim_{n\to\infty}(AX_nu_n, X_nu_n) = -\alpha$. Since α and $-\alpha \in W_0(A)$, it implies that $0 \in W_0(A)$. □

Theorem 4. *Let $\|T - A\| \leq \delta$. Then $|C_T - C_A| \leq \frac{(\delta + [\delta^2 + 8\delta] \|T - C_T\|^{\frac{1}{2}})}{2}$ where C_A is the center of mass of operator A. In this sense, the map $A \to C_A$ is continuous in the uniform operator topology.*

Proof.. We let $C_A = 0$, then

$$\begin{aligned}\|A\|^2 &\geq |C_A|^2 + \|A - C_A\|^2 \\ &\geq |C_A|^2 + \|T - C_A\|^2 - 2\delta\|T - C_A\| + \delta^2 \\ &\geq 2|C_A|^2 + \|T\|^2 - 2\delta(\|T\| + |C_A|) + \delta^2 \\ &\geq \|A\|^2 + (2|C_A|^2 - 2\delta|C_A| - 4\delta\|T\|).\end{aligned}$$

Solving for C_A in the last expression on the right,we conclude that $C_A \leq \frac{(\delta+[\delta^2+8\delta\|T\|^{\frac{1}{2}}])}{2}$. □

Lemma 4. *$W_0(A)\bigcap W_0(A+\beta)=\phi$, for any $\beta\in\mathbb{C},\beta=0$.*

Proof.. Let $\alpha\in W_0(A)\bigcap W_0(A+\beta)$. Then $\|A\|+|\lambda|^2+2Re\overline{\lambda}\beta\leq\|A+\lambda\|$ for $\lambda\in\mathbb{C}$, and$\|A+\beta\|^2+|\lambda|^2+2Re\overline{\lambda}\alpha\leq\|A+\beta+\lambda\|^2,\lambda\in\mathbb{C}$. Letting $\lambda=\beta$ in the first inequality, we obtain $\|A+\beta\|^2+|\beta|^2$. Let $\lambda=-\beta$ in the second inequality, we obtain $\|A+\beta\|^2+|\beta|^2-2Re\overline{\beta}\alpha\leq\|A\|^2$. Combining these yields $|\beta|^2\leq 0$, which completes the proof. □

Theorem 5. *Let δ_A be a derivation on $B(H)$. Then,*

$$\|\delta_A\|=\sup\{\|AX-XA\|:X\in B(H),\|X\|=1\}$$
$$=\inf_{\lambda\in\mathbb{C}}2\|A-\lambda\|.$$

Proof.. Since $\|AX-XA\|=\|(A-\lambda)X+X(A-\lambda)\|$
$\leq 2\|A-\lambda\|\|X\|,$
it follows that $\|\delta_T\|\leq\inf_{\lambda\in\mathbb{C}}2\|A-\lambda\|$. On the other hand, $\|A-\lambda\|$ is large for λ large, so $\inf\|A-\lambda\|$ must be taken at some point, say s_0. But $\|A-s_0\|\leq\|(A-s_0)\|$
$\leq\|(A-s_0)+\lambda\|,$
for all $\lambda\in\mathbb{C}$ implies that $0\in W_0(A-s_0)$.
Hence, $\|\delta_A\|=\|\delta_{A-s_0}\|=2\|A-s_0\|$. □

Theorem 6. *Let G be an irreducible C^*-algebra on H. Let $A\in G(H)$. Then*

$$\|\delta_A|G\|=\sup\{\|AX-XA\|:X\in G,\|X\|=1\}$$
$$=\inf_{\lambda\in\mathbb{C}}2\|A-\lambda\|.$$

Proof.. We use the fact that $B(H)$ contains an operator T such that $Tu=u,Tv=-v$ and $\|T\|=1$ for any $u,v\in H$ where $\langle u,v\rangle=0$. However, if G is an irreducible C^*-algebra then there exists a unitary operator $R\in G$ such that $Ru=u$ and $Rv=-v$ whenever $\langle u,v\rangle=0$.The rest of the proof carries over with only trivial modifications which we shall omit. □

Corollary 7. *Let G_B be an irreducible C^*-algebra on the Hilbert space H_β for β in the index set N. Let $G=\sum_\beta\oplus G_\beta$ on $H=\sum_\beta\oplus H_\beta$ where $\|X\|=\sup_\beta\|X_\beta\|$ for $X\in G$. for $X\in G$. Let $A\in B(H)$, and let $\delta_A:G-G$. Then $\|\delta_A\|=\sup\|AX-XA\|:X\in H,\|X\|=1=\inf\{2\|A-N\|:N\in B(G)\}$ where $B(G)$ is the centre of G.*

Proof.. Since $\delta_A:G-G$ then $A=\sum\oplus A_\beta$ where $A\in B(H_\beta)$. For each β choose λ_β such that $\|\delta_{A_\beta}\|=2\|A-\lambda_\beta\|$. Note that the corollary is not true if we hold our conditions on G. For instance let G contains an operator valued 2×2 matrices on $H\oplus H$ of the form $\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}$, where $X\in B(H)$. Then, $\delta_A:G\to G$. Indeed, $\delta_A=\delta_0$, and so $\|\delta_A\|=0$. But, $\inf_{\lambda\in\mathbb{C}}\{\|A-N\|:N\in B(G)\}=1$. □

Lemma 5. *Suppose that neither S nor T is a scalar multiple of the identity. Then $\inf\{\|S-\lambda\|+\|T-\lambda\|\}=\|S-\lambda_0\|+\|T-\lambda_0\|$ if and only if*

$$W_N(S-\lambda_0)\bigcap W_N(-(T-\lambda_0))\neq\phi.$$

Proof.. Let $W_N(S-\lambda_0)\bigcap W_N(-(T-\lambda_0))\neq\phi$. Then

$$\begin{aligned}\|\delta_{ST}\| &= \|\delta_{(S-\lambda_0),(T-\lambda_0)}\|\\ &= \|S-\lambda_0\|+\|T-\lambda_0\|\end{aligned}$$

Since

$$\begin{aligned}\|SK-KT\| &= \|(S-\lambda)K-K(T-\lambda)\|\\ &\leq \|S-\lambda\|+\|T-\lambda\|\\ &\leq \inf_{\lambda\in\mathbb{C}}\{\|S-\lambda\|+\|T-\lambda\|\}\end{aligned}$$

hence the necessity is shown.
For sufficiency, we assume without loss of generality that $\lambda_0=0$. This means there is $\lambda,\varepsilon\geq 0$ such that there exists $u,v\in H$ of unit norm, so that $\|(S+\lambda)u\|+\|(T+\lambda)v\|\geq\|S\|+\|T\|-\varepsilon$. After some algebra, we find that $Re\overline{\lambda}[(Su,u)/\|S\|+(Tv,v)/\|T\|]\leq B(|\lambda|^2+\varepsilon)$ where B is a constant, independent of λ and ε. Suppose $W_N(S)\bigcap W_N(-T)\neq\phi$.T hen the distinct $[W_N(S),W_N(-T)]=\delta>0$ and by continuity, $dist[W_N(S+\lambda),W_N(-(T+\lambda))]>\frac{\delta}{2}$, for small λ. Thus by convexity and continuity, any choice of u,v which satisfies the above conditions, must satisfy the inequality $|(Su,u)/\|S\|+(Tv,v)/\|T\||\geq\frac{\delta}{4}$ for λ small. But then we are lead to the inequality $|\lambda|\leq B(|\lambda|^2+\varepsilon)$ for a suitable choice of $\arg\lambda$ and a small $|\lambda|$, which is impossible. Thus it is a contradiction since λ was not minimal, hence the proof. □

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