Lie Symmetry Analysis and the Exact Analytic Solutions of Nonlinear Fourth Order Evolution Equation

Kevin. O. Wambura[1], M. E. Oduor Okoya[2], T. J. O. Aminer[3]

[1,2,3]School of Mathematics and Actuarial Science, Jaramogi Oginga Odinga University of Science and Technology, P.O Box 210-40601, Bondo, Kenya

[1]kevinho.junior@yahoo.com

Abstract– **In this paper, a nonlinear fourth order evolution equation is investigated by the Lie symmetry analysis approach. All the geometric vector fields and the Lie groups of the evolution equation are obtained. Finally, the symmetry reduction and the exact solutions of the equation are obtained by means of power series method.**

Keywords– **Evolution Equation, Lie Symmetry Analysis, Vector Fields, Lie Brackets, Lie Groups, Symmetry Reduction and Power Series**

I. INTRODUCTION

Problems involving nonlinear differential equations arise in various fields of science, mathematics and other related areas. Therefore, the task of obtaining the exact solutions of such types of differential equations is of great importance. The theory of Lie symmetry group of differential equations, developed by Sophus Lie, has played a significant role in understanding and constructing solutions of differential equations.

For any given subgroup, an original differential equation can be reduced to a system with fewer independent variables which corresponds to group invariant solutions. In [5], Hongwei et al. discussed the Lie symmetries and the discrete symmetries of the Inviscid Burgers equation. By employing the Lie group method of infinitesimal transformations, they gave the symmetry reductions and similarity solutions of the equation. Based on the discrete analysis, they obtained two groups of discrete symmetry which lead to exact solutions of the inviscid Burgers equation. Bruzon [1] analyzed a general Boussinesq equation using the theory of symmetry reduction of partial differential equations.

The Lie symmetry group analysis of this equation showed that it has only a two-parameter point symmetry group corresponding to traveling wave solutions. To obtain the exact solutions, he used the symmetry reduction to reduce the original nonlinear PDE to a nonlinear ODE and thereafter used the direct method and the (G'/G)-expansion method to arrive at the new solutions of the equation. Moreover, he expressed the traveling wave solutions in terms of the hyperbolic, trigonometric and rational functions. Zhang [14], in his work used Lie symmetry analysis in determining the exact solutions of the Sharma-Tasso-Olever (STO) equation. He obtained the vector fields of the equation, all the similarity reductions and subsequently investigated the exact solutions to the equation by means of power series method. Zhang on the other hand used similar approach in [15], [13] to obtain the exact solutions of Sawada –Kotera equation and seventh-order KdV types of equation respectively.

The solutions of other nonlinear PDEs using Lie symmetry analysis method can also be found in [2], [3], [4], [7], [8], [11].

In this paper, we have investigated the Lie groups, symmetry reductions and the exact solutions of the nonlinear fourth order evolution equation:

$$u_t - 2u_x u_{xx} - u^2 u_{xx} + u_{xxxx} = 0 \qquad (1)$$

using lie symmetry analysis approach.

II. LIE SYMMETRY AND THE GEOMETRIC VECTOR FIELDS

A) Geometric vector fields of Eq. (1)

We let $\Delta = u_t - 2u_x u_{xx} - u^2 u_{xx} + u_{xxxx}$ and the infinitesimal generator X of (1) to be of the form:

$$X = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u} \qquad (2)$$

where the coefficient functions $\tau(x,t,u)$, $\xi(x,t,u)$, and $\eta(x,t,u)$ are to be determined.

For the symmetry condition to be satisfied by (1), then: $X^{(4)}\Delta\big|_{\Delta=0} = 0$.

Here , $X^{(4)}$ is the fourth prolongation of (2) .

By Lie symmetry analysis, the following vector fields are obtained:

$$X_1 = x\frac{\partial}{\partial x} + 4t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u},\quad X_2 = \frac{\partial}{\partial x},\quad X_3 = \frac{\partial}{\partial t}$$

Based on the adjoint representations of the vector fields, we obtain the following optimal systems for Eq. (1) as:
$\{X_1, X_2, X_3, X_3+\lambda X_2\}$, (see [12])
where $\lambda \neq 0$ is an arbitrary constant.

B) Lie groups admitted by Eq. (1)

The one parameter groups G_i admitted by Eq. (1) are determined by solving the corresponding Lie equations below:

$$X_1 : \frac{dx^*}{d\varepsilon} = x^*,\ \frac{dt^*}{d\varepsilon} = 4t^*,\ \frac{du^*}{d\varepsilon} = -u^*$$

$$X_2 : \frac{dx^*}{d\varepsilon} = 1,\ \frac{dt^*}{d\varepsilon} = 0,\ \frac{du^*}{d\varepsilon} = 0$$

$$X_3 : \frac{dx^*}{d\varepsilon} = 0,\ \frac{dt^*}{d\varepsilon} = 1,\ \frac{du^*}{d\varepsilon} = 0$$

with the initial conditions that $x^*_{\varepsilon=0} = x$, $t^*_{\varepsilon=0} = t$ and $u^*_{\varepsilon=0} = u$. This leads to

$$G_1 : (x,t,u;\varepsilon) \to (xe^{\varepsilon}, te^{4\varepsilon}, ue^{-\varepsilon})$$

$$G_2 : (x,t,u;\varepsilon) \to (x+\varepsilon, t, u) \qquad (3)$$

$$G_3 : (x,t,u;\varepsilon) \to (x, t+\varepsilon, u)$$

C) The symmetry reductions of Eq. (1)

One of the main reasons for determining the symmetries of a differential equation is to use them in obtaining symmetry reductions and finding the exact solutions. Therefore we make use of the vector fields X_1, X_2, X_3 and $X_3+\lambda X_2$ to reduce Eq. (1) to systems of ordinary differential equations (ODEs).

The similarity variables and the symmetry reductions of Eq. (1) can be obtained by solving the characteristic equation given by:

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta}. \qquad (4)$$

(a) For the vector field X_1, we have

$$\frac{dx}{x} = \frac{dt}{4t} = \frac{du}{-u}$$

with the invariant z taking the form $z = xt^{-\frac{1}{4}}$.
Therefore u is expressed as :

$$u = t^{-\frac{1}{4}}\psi(z) \qquad (5)$$

Differentiating Eq. (5) and substituting into Eq. (1), we obtain the fourth order ODE given by:

$$-\frac{1}{4}\psi - \frac{1}{4}z\psi' - 2\psi'\psi'' - \psi^2\psi'' + \psi'''' = 0 \qquad (6)$$

where $\psi' = \frac{d\psi}{dz}$.

(b) For the generator X_2, we have

$$\frac{dx}{1} = \frac{dt}{0} = \frac{du}{0}$$

The invariant $z = t$, satisfies

$$u = \psi(z) \qquad (7)$$

Differentiating Eq. (7) and substituting in Eq. (1), we obtain a trivial solution given by:

$$u(x,t) = c \qquad (8)$$

where c is an arbitrary constant.

(c) For the generator X_3, we have

$$\frac{dx}{0} = \frac{dt}{1} = \frac{du}{0}$$

with the invariant z taking the form $z = x$ and u given by :

$$u = \psi(z) \qquad (9)$$

Differentiating Eq. (9) and substituting in Eq. (1), we arrive at

$$-2\psi'\psi'' - \psi^2\psi'' + \psi'''' = 0 \qquad (10)$$

where $\psi' = \frac{d\psi}{dz}$.

(d) For the linear combination $X_3 + \lambda X_2$, we have

$\frac{dx}{\lambda} = \frac{dt}{1} = \frac{du}{0}$. The invariant z is given by $z = x - \lambda t$, where $\lambda > 0$ is the wave velocity. On the other hand, u takes the form:

$$u = \psi(z) \qquad (11)$$

Differentiating Eq. (11) and substituting in Eq. (1), we obtain:

$$-\lambda\psi' - 2\psi'\psi'' - \psi^2\psi'' + \psi'''' = 0 \qquad (12)$$

where $\psi' = \frac{d\psi}{dz}$.

III. THE EXACT SOLUTIONS OF EQ. (1)

Here we consider the exact analytic solutions to the reduced equations by the power series method. Once we obtain the exact analytic solutions of the ordinary differential equations, then the exact power series solutions of the original Partial differential equation (1) are obtained.

A) Exact power series solution of Eq. (6)

We seek a solution in power series of the form:

$$\psi(z)=\sum_{n=0}^{\infty}c_n z^n \tag{13}$$

Differentiating Eq. (13) and substituting in Eq. (6), we have

$$-\frac{1}{4}\sum_{n=0}^{\infty}c_n z^n-\frac{1}{4}\sum_{n=0}^{\infty}(n+1)c_{n+1}z^{n+1}$$
$$-2\left(\sum_{n=0}^{\infty}(n+1)c_{n+1}z^n\right)\left(\sum_{n=0}^{\infty}(n+1)(n+2)c_{n+2}z^n\right)$$
$$-\left(\sum_{n=0}^{\infty}c_n z^n\right)\left(\sum_{n=0}^{\infty}c_n z^n\right)\left(\sum_{n=0}^{\infty}(n+1)(n+2)c_{n+2}z^n\right)$$
$$+\left(\sum_{n=0}^{\infty}(n+1)(n+2)(n+3)(n+4)c_{n+4}z^n\right)=0 \tag{14}$$

On relabeling Eq. (14), we obtain

$$-\frac{1}{4}c_0-\frac{1}{4}\sum_{n=1}^{\infty}c_n z^n-\frac{1}{4}\sum_{n=1}^{\infty}nc_n z^n-4c_1c_2$$
$$-2\sum_{n=1}^{\infty}\sum_{k=0}^{n}(k+1)(n+1-k)(n+2-k)c_{k+1}c_{n+2-k}z^n-2c_0^2c_2$$
$$-\sum_{n=1}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}(n+1-k)(n+2-k)c_ic_{k-i}c_{n+2-k}z^n+24c_4$$
$$+\sum_{n=1}^{\infty}(n+1)(n+2)(n+3)(n+4)c_{n+4}z^n=0 \tag{15}$$

Collecting the terms with similar powers of z together, we have:

$$-\frac{1}{4}c_0-4c_1c_2-2c_0^2c_2+24c_4+\sum_{n=1}^{\infty}\left(-\frac{1}{4}c_n-\frac{1}{4}nc_n\right.$$
$$-2\sum_{k=0}^{n}(k+1)(n+1-k)(n+2-k)c_{k+1}c_{n+2-k}$$
$$-\sum_{k=0}^{n}\sum_{i=0}^{k}(n+1-k)(n+2-k)c_ic_{k-i}c_{n+2-k}$$
$$\left.+(n+1)(n+2)(n+3)(n+4)c_{n+4}\right)z^n=0 \tag{16}$$

From Eq. (16), we have that for $n=0$,

$$c_4=\frac{c_0}{96}+\frac{c_1c_2}{6}+\frac{c_0^2c_2}{12} \tag{17}$$

For $n\geq 1$, we obtain the recurrence relation:

$$c_{n+4}=\frac{1}{(n+1)(n+2)(n+3)(n+4)}\left(\frac{c_n}{4}+\frac{nc_n}{4}\right.$$
$$+2\sum_{k=0}^{n}(k+1)(n+1-k)(n+2-k)c_{k+1}c_{n+2-k}$$
$$\left.+\sum_{k=0}^{n}\sum_{i=0}^{k}(n+1-k)(n+2-k)c_ic_{k-i}c_{n+2-k}\right) \tag{18}$$

where $c_i(i=1,2,.....)$ are arbitrary constants.

For $n=1,2$ we obtain;

$$c_5=\frac{1}{60}\left(\frac{c_1}{2}+6c_1c_3+4c_2^2+3c_0^2c_3+2c_0c_1c_2\right)$$
$$c_6=\frac{1}{360}\left(\frac{3}{4}c_2+24c_1c_4+36c_2c_3+12c_0^2c_4\right.$$
$$\left.+12c_0c_1c_3+4c_0c_2^2+2c_1^2c_2\right)$$

Hence the power series solution of Eq. (6) can expressed as :

$$\psi(z)=c_0+c_1z+c_2z^2+c_3z^3+\sum_{n=1}^{\infty}c_{n+4}z^{n+4}$$
$$=c_0+c_1z+c_2z^2+\left(\frac{c_0}{96}+\frac{c_1c_2}{6}+\frac{c_0^2c_2}{12}\right)z^4$$
$$+\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)(n+3)(n+4)}\left(\frac{c_n}{4}+\frac{nc_n}{4}\right.$$
$$+2\sum_{k=0}^{n}(k+1)(n+1-k)(n+2-k)c_{k+1}c_{n+2-k}$$
$$\left.-\sum_{k=0}^{n}\sum_{i=0}^{k}(n+1-k)(n+2-k)c_ic_{k-i}c_{n+2-k}\right)z^{n+4}$$

Thus the exact power series solution of Eq. (1) is:

$$u(x,t)=c_0t^{-\frac{1}{4}}+c_1xt^{-\frac{1}{2}}+c_2x^2t^{-\frac{3}{4}}+c_3x^3t^{-1}$$
$$+\left(\frac{c_0}{96}+\frac{c_1c_2}{6}+\frac{c_0^2c_2}{12}\right)x^4t^{-\frac{5}{4}}$$
$$+\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)(n+3)(n+4)}\left(\frac{c_n}{4}+\frac{nc_n}{4}\right.$$
$$+2\sum_{k=0}^{n}(k+1)(n+1-k)(n+2-k)c_{k+1}c_{n+2-k}$$
$$\left.+\sum_{k=0}^{n}\sum_{i=0}^{k}(n+1-k)(n+2-k)c_ic_{k-i}c_{n+2-k}\right)x^{n+4}t^{-\frac{n+5}{4}} \tag{19}$$

According to [14], the solution to partial differential equations, in mathematics and physical applications, can conveniently be expressed in approximate form. Thus, the solution of Eq. (1) can be expressed as:

$$
\begin{aligned}
u(x,t) &= c_0 t^{-\frac{1}{4}} + c_1 x t^{-\frac{1}{2}} + c_2 x^2 t^{-\frac{3}{4}} + c_3 x^3 t^{-1} \\
&+ \left(\frac{c_0}{96} + \frac{c_1 c_2}{6} + \frac{c_0^2 c_2}{12} \right) x^4 t^{-\frac{5}{4}} \\
&+ \frac{1}{60}\left(\frac{c_1}{2} + 6c_1c_3 + 4c_2^2 + 3c_0^2 c_3 + 2c_0c_1c_2 \right) x^5 t^{-\frac{3}{2}} \\
&+ \frac{1}{360}\left(\frac{3}{4}c_2 + 24c_1c_4 + 36c_2c_3 + 12c_0^2 c_4 \right. \\
&\left. + 12c_0c_1c_3 + 4c_0c_2^2 + 2c_1^2c_2 \right) x^6 t^{-\frac{7}{4}} + ...
\end{aligned} \tag{20}
$$

B) The stationary solution to Eq. (10)

Here we seek a solution of Eq. (10) in power series of the form of Eq. (13). Substituting Eq. (13) into Eq. (10), we have

$$
\begin{aligned}
&-2\left(\sum_{n=0}^{\infty} (n+1) c_{n+1} z^n \right)\left(\sum_{n=0}^{\infty} (n+1)(n+2) c_{n+2} z^n \right) \\
&-\left(\sum_{n=0}^{\infty} c_n z^n \right)\left(\sum_{n=0}^{\infty} c_n z^n \right)\left(\sum_{n=0}^{\infty} (n+1)(n+2) c_{n+2} z^n \right) \\
&+\left(\sum_{n=0}^{\infty} (n+1)(n+2)(n+3)(n+4) c_{n+4} z^n \right) = 0
\end{aligned} \tag{21}
$$

Relabeling Eq. (21) and collecting the terms with similar powers of z together, we obtain:

$$
\begin{aligned}
&-4c_1c_2 - 2c_0^2c_2 + 24c_4 \\
&+\sum_{n=1}^{\infty}\left\{ -2\sum_{k=0}^{\infty}(k+1)(n+1-k)(n+2-k)c_{k+1}c_{n+2-k} \right. \\
&-\sum_{k=0}^{n}\sum_{i=0}^{k}(n+1-k)(n+2-k)c_i c_{k-i} c_{n+2-k} \\
&\left. +(n+1)(n+2)(n+3)(n+4)c_{n+4} \right\} z^n = 0
\end{aligned} \tag{22}
$$

From Eq. (22), we have that for $n = 0$,

$$
c_4 = \frac{c_1c_2}{6} + \frac{c_0^2c_2}{12} \tag{23}
$$

For $n \geq 1$, we obtain the recurrence relation:

$$
\begin{aligned}
c_{n+4} = &\frac{1}{(n+1)(n+2)(n+3)(n+4)}\left\{ 2\sum_{k=0}^{n}(k+1) \right. \\
&\left(n+1-k\right)\left(n+2-k\right)c_{k+1}c_{n+2-k} \\
&\left. +\sum_{k=0}^{n}\sum_{i=0}^{k}\left(n+1-k\right)\left(n+2-k\right)c_i c_{k-i} c_{n+2-k} \right\}
\end{aligned} \tag{24}
$$

for all $n = 1, 2, ...$

Thus, for arbitrary chosen constants $c_i \left(i = 0,1,2,3\right)$, we obtain:

$$
c_5 = \frac{1}{120}\left(12c_1c_3 + 8c_2^2 + 6c_0^2c_3 + 4c_0c_1c_2\right)
$$

$$
c_6 = \frac{1}{360}\left(24c_1c_4 + 36c_2c_3 + 12c_0^2c_4 + 12c_0c_1c_3 + 4c_0c_2^2 + 2c_1^2c_2\right)
$$

and so on.

Hence the power series solution Eq. (10) can be expressed as:

$$
\begin{aligned}
\psi(z) &= c_0 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \sum_{n=1}^{\infty} c_{n+4} z^{n+4} \\
&= c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \left(\frac{c_1c_2}{6} + \frac{c_0^2c_2}{12} \right) z^4 \\
&+ \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)(n+3)(n+4)} \\
&\times \left(2\sum_{k=0}^{n}(k+1)(n+1-k)(n+2-k)c_{k+1}c_{n+2-k} \right. \\
&\left. + \sum_{k=0}^{n}\sum_{i=0}^{k}(n+1-k)(n+2-k)c_i c_{k-i} c_{n+2-k} \right) z^{n+4}
\end{aligned} \tag{25}
$$

Thus, the exact stationary solution to Eq. (1) is given as:

$$
\begin{aligned}
u(x,t) &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \left(\frac{c_1c_2}{6} + \frac{c_0^2c_2}{12} \right) x^4 \\
&+ \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)(n+3)(n+4)} \\
&\times \left(2\sum_{k=0}^{n}(k+1)(n+1-k)(n+2-k)c_{k+1}c_{n+2-k} \right. \\
&\left. + \sum_{k=0}^{n}\sum_{i=0}^{k}(n+1-k)(n+2-k)c_i c_{k-i} c_{n+2-k} \right) x^{n+4}
\end{aligned} \tag{26}
$$

C) The traveling wave solution for Eq. (12)

Substituting Eq. (13) into Eq. (12), he have:

$$-\lambda\sum_{n=0}^{\infty}(n+1)c_{n+1}z^n$$
$$-2\left(\sum_{n=0}^{\infty}(n+1)c_{n+1}z^n\right)\left(\sum_{n=0}^{\infty}(n+1)(n+2)c_{n+2}z^n\right)$$
$$-\left(\sum_{n=0}^{\infty}c_nz^n\right)\left(\sum_{n=0}^{\infty}c_nz^n\right)\left(\sum_{n=0}^{\infty}(n+1)(n+2)c_{n+2}z^n\right)$$
$$+\left(\sum_{n=0}^{\infty}(n+1)(n+2)(n+3)(n+4)c_{n+4}z^n\right)=0 \tag{27}$$

On relabeling Eq. (27) and collecting the terms with similar powers of z together, we obtain:

$$-\lambda c_1-4c_1c_2-2c_0^2c_2+24c_4$$
$$-\sum_{n=1}^{\infty}\Big\{\lambda(n+1)c_{n+1}+2\sum_{k=0}^{n}(k+1)(n+1-k)(n+2-k)c_{k+1}c_{n+2-k}$$
$$+\sum_{k=0}^{n}\sum_{i=0}^{k}(n+1-k)(n+2-k)c_ic_{k-i}c_{n+2-k}$$
$$-(n+1)(n+2)(n+3)(n+4)c_{n+4}\Big\}z^n=0 \tag{28}$$

From Eq. (28), we have that for $n=0$;

$$c_4=\frac{\lambda c_1}{24}+\frac{c_1c_2}{6}+\frac{c_0^2c_2}{12} \tag{29}$$

For $n\geq 1$, we obtain the following recurrence relation:

$$c_{n+4}=\frac{1}{(n+1)(n+2)(n+3)(n+4)}\Big\{\lambda(n+1)c_{n+1}$$
$$+2\sum_{k=0}^{n}(k+1)(n+1-k)(n+2-k)c_{k+1}c_{n+2-k}$$
$$+\sum_{k=0}^{n}\sum_{i=0}^{k}(n+1-k)(n+2-k)c_ic_{k-i}c_{n+2-k}\Big\} \tag{30}$$

Thus, for arbitrary chosen constants $c_i\,(i=0,1,2,3)$ we obtain:

$$c_5=\frac{1}{120}\left(2\lambda c_2+12c_1c_2+8c_2^2+6c_0^2c_3+4c_0c_1c_3\right)$$
$$c_6=\frac{1}{360}\Big(3\lambda c_3+24c_1c_4+36c_2c_3+12c_0^2c_4$$
$$+12c_0c_1c_3+4c_0c_2^2+2c_1^2c_2\Big) \tag{31}$$

and so on.

Hence the power series solution of Eq. (12) can be expressed as:

$$\psi(z)=c_0+c_1z+c_2z^2+c_3z^3+c_4z^4+\sum_{n=1}^{\infty}c_{n+4}z^{n+4}$$
$$=c_0+c_1z+c_2z^2+c_3z^3+\left(\frac{\lambda c_1}{24}+\frac{c_1c_2}{6}+\frac{c_0^2c_2}{12}\right)z^4$$
$$+\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)(n+3)(n+4)}\Big\{\lambda(n+1)c_{n+1}$$
$$+2\sum_{k=0}^{n}(k+1)(n+1-k)(n+2-k)c_{k+1}c_{n+2-k}$$
$$+\sum_{k=0}^{n}\sum_{i=0}^{k}(n+1-k)(n+2-k)c_ic_{k-i}c_{n+2-k}\Big\}z^{n+4} \tag{32}$$

Therefore, the exact traveling wave solution to Eq. (1) is given as:

$$u(x,t)=c_0+c_1(x-\lambda t)+c_2(x-\lambda t)^2+c_3(x-\lambda t)^3$$
$$+\left(\frac{\lambda c_1}{24}+\frac{c_1c_2}{6}+\frac{c_0^2c_2}{12}\right)(x-\lambda t)^4$$
$$+\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)(n+3)(n+4)}\Big\{\lambda(n+1)c_{n+1}$$
$$+2\sum_{k=0}^{n}(k+1)(n+1-k)(n+2-k)c_{k+1}c_{n+2-k}$$
$$+\sum_{k=0}^{n}\sum_{i=0}^{k}(n+1-k)(n+2-k)c_ic_{k-i}c_{n+2-k}\Big\}(x-\lambda t)^{n+4} \tag{33}$$

Note: Power series is a useful and powerful method for solving higher order nonlinear ordinary differential equations (ODE). Therefore, the power series solutions (19), (26) and (33) are the exact analytic solutions to Eq. (1).

Remark*:* According to Olver P.J [10], if $u=\psi(x,t)$ is a solution of Eq. (1), so are the functions:

$$G_1(\varepsilon)\psi(x,t)=\psi(xe^{-\varepsilon},te^{-4\varepsilon})e^{-\varepsilon}$$
$$G_2(\varepsilon)\psi(x,t)=\psi(x-\varepsilon,t) \tag{34}$$
$$G_3(\varepsilon)\psi(x,t)=\psi(x,t-\varepsilon)$$

That is, a symmetry group of Eq. (1) is a local group of transformation G with the property that whenever $u=\psi(x)$ is a solution of Eq. (1) and whenever $g.\psi$ is defined for $g\in G$, then $u=g.\psi$ is also a solution of Eq. (1). Thus, there is a possibility of constructing a whole family of solutions just by transforming a known solution by all possible group elements.

Therefore, using G_1, the solutions (19), (26) and (33) can respectively be expressed as:

$$u^*(x^*,t^*)=\Big\{c_0(t^*e^{-4\varepsilon})^{-\frac{1}{4}}+c_1(x^*e^{-\varepsilon})(t^*e^{-4\varepsilon})^{-\frac{1}{2}}$$
$$+c_2(x^*e^{-\varepsilon})^2(t^*e^{-4\varepsilon})^{-\frac{3}{4}}+c_3(x^*e^{-\varepsilon})^3(t^*e^{-4\varepsilon})^{-1}$$
$$+\left(\frac{c_0}{96}+\frac{c_1c_2}{6}+\frac{c_0^2c_2}{12}\right)(x^*e^{-\varepsilon})^4(t^*e^{-4\varepsilon})^{-\frac{5}{4}}$$
$$+\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)(n+3)(n+4)}\left(\frac{c_n}{4}+\frac{nc_n}{4}\right.$$
$$+2\sum_{k=0}^{n}(k+1)(n+1-k)(n+2-k)c_{k+1}c_{n+2-k}$$
$$\left.+\sum_{k=0}^{n}\sum_{i=0}^{k}(n+1-k)(n+2-k)c_ic_{k-i}\,c_{n+2-k}\right)$$
$$(x^*e^{-\varepsilon})^{n+4}(t^*e^{-4\varepsilon})^{-\frac{n+5}{4}}\Big\}e^{-\varepsilon} \tag{35}$$

$$u^*(x^*,t^*)=\Big\{c_0+c_1(x^*e^{-\varepsilon})+c_2(x^*e^{-\varepsilon})^2$$
$$+c_3(x^*e^{-\varepsilon})^3+\left(\frac{c_1c_2}{6}+\frac{c_0^2c_2}{12}\right)(x^*e^{-\varepsilon})^4$$
$$+\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)(n+3)(n+4)}$$
$$\times\left(2\sum_{k=0}^{n}(k+1)(n+1-k)(n+2-k)c_{k+1}c_{n+2-k}\right.$$
$$\left.+\sum_{k=0}^{n}\sum_{i=0}^{k}(n+1-k)(n+2-k)c_ic_{k-i}\,c_{n+2-k}\right)$$
$$(x^*e^{-\varepsilon})^{n+4}\Big\}e^{-\varepsilon} \tag{36}$$

$$u^*(x^*,t^*)=\Big\{c_0+c_1(x^*e^{-\varepsilon}-\lambda t^*e^{-4\varepsilon})$$
$$+c_2\left(x^*e^{-\varepsilon}-\lambda t^*e^{-4\varepsilon}\right)^2+c_3\left(x^*e^{-\varepsilon}-\lambda t^*e^{-4\varepsilon}\right)^3$$
$$+\left(\frac{\lambda c_1}{24}+\frac{c_1c_2}{6}+\frac{c_0^2c_2}{12}\right)\left(x^*e^{-\varepsilon}-\lambda t^*e^{-4\varepsilon}\right)^4$$
$$+\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)(n+3)(n+4)}\Big\{\lambda(n+1)c_{n+1}$$
$$+2\sum_{k=0}^{n}(k+1)(n+1-k)(n+2-k)c_{k+1}c_{n+2-k}$$
$$+\sum_{k=0}^{n}\sum_{i=0}^{k}(n+1-k)(n+2-k)c_ic_{k-i}\,c_{n+2-k}\Big\}$$
$$\left(x^*e^{-\varepsilon}-\lambda t^*e^{-4\varepsilon}\right)^{n+4}\Big\}e^{-\varepsilon} \tag{37}$$

G_2 and G_3 can also be used in a similar manner.

IV. CONCLUSION

In this paper, we have obtained the geometric vector fields, Lie groups and the symmetry reduction of the nonlinear fourth order evolution equation (1) using Lie symmetry analysis method. Moreover, all the group invariant solutions to the equation have been considered based on the optimal system method and the exact analytic solutions to the equation investigated using power series approach. Finally, we have shown that a whole family of solutions can be constructed just by transforming a known solution by all the possible group elements.

REFERENCES

[1] Bruzon M.S (2009), Exact solutions of the generalized Boussinesq equation, *Theoretical and mathematical physics*; 160(1), 894-904

[2] Gangwei W, Tianzhou X (2015), Invariant analysis and explicit solutions of the time fractional nonlinear perturbed Burgers equation, *Nonlinear analysis: modelling and control*, Vol. 20, No. 4, 570-584

[3] Hanze l, Jibin L, Quanxin Z (2008), Lie symmetry analysis and the exact explicit solutions to the general Burgers equation, *J. Math. Anal.* 228, 1-9

[4] Hanze L, Jibin L, Lei L (2010), Lie symmetry analysis, optimal systems and the exact solutions to the fifth –order KdV types of equations, *J.Math. Anal. App 368*, 551-558.

[5] Hongwei Y, Yunglong S, Baoshu Y, Huanhe D (2012), Discrete symmetries analysis and exact solutions of the Inviscid Burgers equation, *Hindawi publishing corporation*, ID 908975

[6] Kadkhoda N, Jafari H, Morenedi G.M, Baleanu D (2016), Optimal systems and symmetry reduction of the (1+1) dimensional Sawada-Kotera equation, *International Journal of Pure and Applied mathematics*, Vol. 103, No. 2, 215-226

[7] Mehdi N, Vahid S (2011), Lie symmetry analysis of Kudryashov-Sinelshchilov equation, *Hindawi publishing corporation*, ID 457697.

[8] Motsepa T (2016), Symmetry analysis, conservation laws and exact solutions of certain nonlinear partial differential equation, *PhD thesis, Mafikeng Campus of North-west India*.

[9] Oduor Okoya M.E, Ongati N.O, Aminer T.J.O, Nyakinda J.O (2013), Symmetry group approach to the solution of generalized Burgers equation, *Applied mathematical Sciences*, Vol. 7, No. 15, 717-725.

[10] Olver P.J (1986), Application of Lie groups to differential equations, *Springer-Verlag, New York , NY, USA.*

[11] Spirana N, Chatanin W (2016), Lie symmetry analysis and exact solutions to the Quintin nonlinear Beam equation, *Malaysian Journal of mathematical Sciences*, 10(1), 61-68.

[12] Wambura K.O, Oduor M.E, Aminer T.J.O (2019), Lie symmetry analysis and the optimal systems of Nonlinear fourth order evolution equation, *IJSEAS*, Vol. 5, Issue-3, 1-6.

[13] Zhang Y (2015), Symmetry reduction and exact solutions to the seventh-order KdV types of equations, *Konuralp Journal of Mathematics*, Vol. 3, No. 2, 122-130.

[14] Zhang Y (2016), Lie symmetry analysis and the exact solutions of the Sharma-Tasso-Olever equation, *International Journal of Applied Mathematics*, 46:2

[15] Zhang Y (2017), Lie symmetry analysis and the exact solutions of the Sawada-Kotera equation, *Turk J. Math* 41, 158-167.