# Creating a New Nest Algebra Structure for Nilpotent Lie Algebras

**Atefeh Hasan-Zadeh**
Fouman Faculty of Engineering, College of Engineering, University of Tehran, Iran

***Abstract*—In this paper a functional analysis approach to solve the rigidity problems of the Riemannian nilmanifolds using information from their geodetic flows is considered. The proposed method combines analytical and geometric materials to construct a structure of nest algebra for nilpotent Lie algebras. In this way, the main concepts of Lie derivations and the results obtained from it in nilmanifolds are consistent with the corresponding results in nest algebras. Due to this adaptation of the main concepts, one proposition of Gordon, which can be used in solving the rigidity problem for 2-step nilmanifolds, has been proved in the new structure. Also, the resulting nesting algebra structure leads to an extension of that proposition to quasi-nilpotent operators.**

***Keywords*— Nilpotent Lie Algebra, Inner Derivations, Trace Class Operators and Quasi-nilpotent Operators**

## I. INTRODUCTION

IN recent years, considerable attention has been paid to solving the rigidity problems for nilmanifolds via some information about their geodesic flows:

**Problem.** ([2]) Whether two compact 2-step nilmanifolds $\frac{M}{\Gamma}$ and $\frac{M'}{\Gamma'}$ are isometric or not, if they have conjugated geodesic flows?

Among which one can be mentioned through the works of Eberlein, Gordon, Mao and Schueth in [1-3]. This problem has been already considered in [4] by an algebraic-geometric approach, especially in the category of Lie groupoids, and in [5] in notions of von Neumann algebras. Also, the symplectic version of it, [3], has been extended to Poisson notions, [6].

In our ongoing research on this issue, we have come up with an analytical approach. The method presented in this paper uses the structure of nest algebras corresponding to the algebra of projections on complex Hilbert spaces. In this way, by some ingredients of operator algebra and geometry, we construct a structure of nest algebra for an $m$-dimensional nilpotent Lie algebra, Theorem 3.1. The structure obtained is such that the main concepts of Lie derivations and the results obtained from it in nilmanifolds are consistent with the corresponding results in nest algebras. For this reason, we could prove one proposition of Gordon, which can be used in solving the rigidity problem for 2-step nilmanifolds, in a new structure, Corollary 3.1. It should be noted that this proposition was proved once again in [4] with the use of the concepts of Lie groupoids. Lastly, in expressing the importance of the approach used, an extension of Corollary 3.1 to quasi-nilpotent operators has been mentioned in Corollary 3.2.

## II. PRELIMINARIES

### A. *Ingredients of operator algebra*

Let $\mathcal{H}$ be a complex Hilbert space and $B(\mathcal{H})$ is the algebra of all linear bounded operators on $\mathcal{H}$. A nest $\mathcal{N}$ of projections on $\mathcal{H}$ is a chain of orthogonal projections on $\mathcal{H}$ which contains $0$, $I$ and is closed in the strong operator topology. The nest algebra $\mathcal{T}(\mathcal{N})$ corresponding to the nest $\mathcal{N}$ is the set of all operators $A$ in such that $AP = PA$ for all $P \in \mathcal{N}$. Clearly, $\mathcal{T}(\mathcal{N})$ is a weak* closed operator algebra. For more details refer to [7-9].

**Theorem 2.1.** ([7]) Let $\mathcal{N}$ be a nest of projections on a complex Hilbert space $\mathcal{H}$ and $\mathcal{B}$ be a weak* closed subalgebra of $B(\mathcal{H})$ which contains $\mathcal{T}(\mathcal{N})$. Then a linear map $\delta : \mathcal{T}(\mathcal{N}) \to \mathcal{B}$ is a Lie triple derivation if and only if there are an element $S \in \mathcal{B}$ and a linear functional $f$ on $\mathcal{T}(\mathcal{N})$ with $f(D) = 0$ for every second commutator $D = [[X,Y],Z]$ such that $\delta(A) = [S, A] + f(A)I$, $\forall A \in \mathcal{T}(\mathcal{N})$.

**Corollary 2.1.** ([7]) Let $\mathcal{N}$ be a nest of infinite multiplicity on a complex Hilbert space $\mathcal{H}$ and let $\mathcal{B}$ be a weak* closed subalgebra of $B(\mathcal{H})$ which contains $\mathcal{T}(\mathcal{N})$. Let $\delta$ be a Lie triple derivation from $\mathcal{T}(\mathcal{N})$ into $\mathcal{B}$. Then there is an element $S \in \mathcal{B}$ such that $\delta(A) = [S, A]$, $\forall A \in \mathcal{T}(\mathcal{N})$.

In fact, Lie derivations are special cases of Lie triple derivations.

A. Hasan-Zadeh is a faculty member of Fouman Faculty of Engineering, College of Engineering, University of Tehran, Iran, P.O..Box: 43581-39115, (e-mail: hasanzadeh.a@ut.ac.ir).

### *B. Ingredients of geometry*

As the notions of [1-3], a Riemannian nilmanifold is a quotient $\frac{M}{\Gamma}$ of a simply connected nilpotent Lie group $M$ by a discrete subgroup $\Gamma$; together with a Riemannian metric $g$ whose lift to $M$ is left invariant. A nilmanifold $\frac{M}{\Gamma}$ has step size $k$ if $M$ is $k$-step nilpotent.

Especially, a Lie group $M$ is said to be two-step nilpotent if its Lie algebra $\mathcal{M}$ satisfies $[[\mathcal{M},\mathcal{M}],\mathcal{M}]=0$, equivalently $[\mathcal{M},\mathcal{M}]$ is central in $\mathcal{M}$. If $g$ be a left-invariant Riemannian metric on 2-step nilpotent Lie group $M$, then it defines an inner product $\langle .,. \rangle$ on the Lie algebra $\mathcal{M}$ of $M$. Let $\mathcal{Z}=[\mathcal{M},\mathcal{M}]$ and $\mathcal{V}$ denote the orthogonal complement of $\mathcal{Z}$ in $\mathcal{M}$ relative to $\langle .,. \rangle$. For $z\in\mathcal{Z}$, a skew symmetric linear transformation $J(z):\mathcal{V}\to\mathcal{V}$ can be defined by $J(z)x=(ad(x))^{*}z$ for $x\in\mathcal{V}$, where $(ad(x))^{*}$ denotes the adjoint of $ad(x)$. Equivalently,

$$\langle J(z)x,y\rangle=\langle [x,y],z\rangle,\forall x,y\in\mathcal{V},z\in\mathcal{Z} \quad (2.1)$$

and this process is reversible.

An authomorphism $\Phi$ of $M$ is said to be $\Gamma$-almost inner if $\Phi(\gamma)$ is conjugate to $\gamma$ for all $\gamma\in\Gamma$. It is said to be almost inner if $\Phi(x)$ is conjugate to $x$ for all $x\in M$. A derivation $\varphi$ of the Lie algebra $\mathcal{M}$ is said to be $\Gamma$-almost inner (respectively, almost inner) if $\varphi(X)\in[\mathcal{M},X]$ for all $X\in\log\Gamma$ (respectively, for all $x\in\mathcal{M}$. The spaces of $\Gamma$-almost inner automorphisms is denoted by $AIA(\Gamma)$ and similarly, $AID(\Gamma)$ is its infinitesimal counterpart, namely $\Gamma$-almost inner derivations.

## III. MAIN STRUCTURE

**Theorem 3.1.** For an m-dimensional nilpotent Lie algebra $\mathcal{M}$, it can be constructed a structure of nest algebra $\mathcal{T}(\mathcal{M})$ corresponded to it which the notions of Lie derivations in these two apparently different structures produce the same meanings and results (especially Theorem 2.1 and Corollary 2.1 in this structure will also be satisfied).

*Proof.* Suppose that $\mathcal{M}$ is an $m$-dimensional nilpotent Lie algebra. Denote by $\mathcal{M}^{*}$ the dual space of $\mathcal{M}$ and $M$ by the connected and simply connected Lie group with Lie algebra $\mathcal{M}$. Let $\mathcal{M}_0\subset\mathcal{M}_1...\subset\mathcal{M}_m=\mathcal{M}$ be a flag of $\mathcal{M}$ ($\dim\mathcal{M}_i=i$) such that $[\mathcal{M},\mathcal{M}_i]\subset\mathcal{M}_{i-1}$ for all $i\in\{1,...,m\}$. Let also $B=(X_1,...,X_m)$ be a Jordan-Holder basis adapted to $(\mathcal{M}_i)_i$ that $\mathcal{M}_i=\mathbf{R}X_1\oplus...\oplus\mathbf{R}X_i$, for all $i$. Then, $\mathcal{M}^{*}$ has a natural layering which can be summarized as follows.

For $\mu$ in $\mathcal{M}^{*}$, it can be defined the set of indexes $J_{\mu}=\left\{j:X_j\notin\left(\mathcal{M}_{i-1}+\mathcal{M}_{\mu}\right)\right\}$, where $\mathcal{M}_{\mu}=\left\{X\in\mathcal{M}:\langle\mu,[X,Y]\rangle=0,\forall Y\in\mathcal{M}\right\}$. If $J_{\mu}=\{j_1<...<j_{2r}\}$ then $\mathcal{M}=\mathcal{M}_{\mu}\oplus\mathbf{R}X_{j_1}\oplus...\oplus\mathbf{R}X_{j_{2r}}$. Let $\Delta=\{J_{\mu};\mu\in\mathcal{M}^{*}\}$. For $e\in\Delta$, it can be defined the layer $\mathcal{M}_B^{e}=\{\mu\in\mathcal{M}^{*}:J_{\mu}=0\}$. By construction, each layer is a $M$-invariant subset of $\mathcal{M}^{*}$ and $\mathcal{M}^{*}$ is a disjoint finite union of layers $\mathcal{M}^{*}=\bigcup_{e\in\Delta}\mathcal{M}^{*e}_B$. Also, all the orbits contained in a given layer have the same dimension ($card\ e$). Let the orbits contained in $\mathcal{M}^{*e}_B$ are $2r$-dimensional. Then, there exists on $\mathcal{M}^{*}$

i) $m-2r$ polynomial functions $z_1,...,z_{m-2r}$;

ii) $2r$ rational functions $p_1,...,p_r,q_1,...,q_r$ which are regular on $\mathcal{M}^{*e}_B$ such that the polynomial functions $z_1,...,z_{m-2r}$ separate the orbits contained in $\mathcal{M}^{*e}_B$;

The first layer has additional properties: it is a Zariski dense open subset of $\mathcal{M}^{*}$, it contains only orbits of maximal dimension $2d$, and the polynomial functions $z_1,...,z_{m-2d}$ separating the orbits of $\mathcal{M}^{*e}_B$ are $M$-invariant.

Moreover, if we identify the symmetric algebra $S(M)$ of $M$ with the space of polynomial functions on $\mathcal{M}^{*}$ and denote by $S(M)^{M}$ the subring of $S(M)$ of the $M$invariant polynomial functions, then the quotient field of $S(M)^{M}$ coincides exactly with the field $R(z_1,...,z_{m-2d})$ of rational functions in the $z_k$ variables. The open set $\mathcal{M}^{*e}_B$ is usually called the generic set associated to the basis $B$, the orbits contained in $\mathcal{M}^{*e}_B$ are the generic orbits and the corresponding polynomial functions $z_1,...,z_{m-2d}$ are the generic invariants.

Let $\tilde{\mathcal{M}}^{*}$ is any open subset of $\mathcal{M}^{*}$ such that the polynomial functions $z_1,...,z_{m-2d}$ separate the orbits contained in $\tilde{\mathcal{M}}^{*}$ and the vectors $dz_1(\mu),...,dz_{m-2d}(\mu)$ are linearly independent for all $\mu$ in $\tilde{\mathcal{M}}^{*}$. Also, since the generic invariants $z_1,...,z_{m-2d}$ associated to any basis $B$ generate the quotient field of $S(M)^{M}$, then $\tilde{\mathcal{M}}^{*}$ satisfies the above conditions for any Jordan-Holder basis.

**Corollary 3.1.** ([2]) Let $\mathcal{M}$ be a 2-step nilpotent Lie algebra with an inner product $\langle .,. \rangle$ and $\varphi$ be an almost inner derivation of continuous type on $\mathcal{M}$ say $\varphi(x)=[\sigma(x),x]$ with $\sigma$ continuous on $\mathcal{M}\setminus\{0\}$. Let $z\in Z(M)$ and $y\in\ker(J(z))$. Then $\langle\varphi(x),z\rangle=\langle[\sigma(x),y],z\rangle$, $\forall x\in\mathcal{M}$, where $J(z):\mathcal{V}\to\mathcal{V}$, is a skew symmetric linear transformation defined by equation (2.1).

In special case, if the center of $\mathcal{M}$ properly contains the derived algebra, then every almost inner derivation of continuous type on $\mathcal{M}$ is inner.

*Proof.* From Theorem 3.1 applied to Theorem 2.1 and Corollary 2.1, it can be concluded that:

A Lie derivation (especially, almost inner derivation) $\varphi$ (especially, $\varphi(x)=[\sigma(x),x]$, on the nest algebra $\mathcal{T}(\mathcal{M})$ where

$S$ is a weak* closed subalgebra which contains $\mathcal{T}(\mathcal{M})$ (especially, $\langle \varphi(x), z\rangle = \langle [\sigma(x), y], z\rangle$, $\forall x \in \mathcal{M}$) and vice versa.

As a result, it has been archived more than we expected.

**Corollary 3.2.** Corollary 3.1 can equally be extended to quasi-nilpotent operators rather than almost inner derivations.

*Proof.* A bounded linear operator $T$ over a (separable) Hilbert space $\mathcal{H}$ is said to be in the trace class if for some (and hence all) orthonormal bases $\{e_k\}_k$ of $\mathcal{H}$, the sum of positive terms $\|A\|_1 = Tr|A| := \sum_k \left\langle (A^*A)^{\frac{1}{2}} e_k, e_k \right\rangle$ is finite. Then it can be defined the trace of $T$, $Tr(T) = \sum_k \langle Ae_k, e_k\rangle$ which is independent of the choice of the orthonormal basis.

On the other hand, as a corollary of Erdos Density Theorem, ([7]), every trace class operator on a Hilbert space $\mathcal{H}$ is the limit in the trace norm of finite rank operators in $\mathcal{T}(\mathcal{N})$. Then every trace class quasi-nilpotent operator on a Hilbert space $\mathcal{H}$ is the limit in the trace norm of finite rank nilpotent operators. Then, Corollary 3.1 can bring the same result for the quasi-nilpotent operators instead of almost inner derivations.

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