The Classical Lie Symmetry Approach to Solution of Modified Type of Burgers Equation

Kevin. O. Wambura[1] and T. J. O. Aminer[2]

[1*,2,] School of Biological, Physical, Mathematics and Actuarial Sciences, Jaramogi Oginga Odinga University of Science and Technology, P.O Box 210-40601, Bondo, Kenya

[*]kevinho.junior@yahoo.com

***Abstract*– In this paper, a modified type of Burgers equation is investigated by the Lie symmetry analysis approach. All the geometric vector fields and the Lie groups admitted by the equation are constructed. Finally the symmetry reduction and the symmetry solutions of the equation are obtained.**

***Index Terms*– Nonlinear PDEs, Lie Symmetry Analysis, Vector Fields, Lie Groups, Symmetry Reduction and Power Series**

I. INTRODUCTION

The investigation of exact solutions of nonlinear PDEs plays an important role in the study of nonlinear physical phenomena for instance in shallow water waves, fluid physics, general relativity and may others. Lie symmetry analysis, pioneered by Sophus Lie, has played a significant role in the construction of exact solutions to nonlinear partial differential equations. The Burgers equation is a fundamental nonlinear PDE occurring in solitary wave theory and various areas of applied mathematics such as fluid mechanics, nonlinear acoustics, gas dynamics and traffic flow.

In [4], Hongwei et al. discussed the Lie symmetry and the discrete symmetries of the Inviscid Burgers equation. By employing the Lie group method of infinitesimal transformation, they gave the symmetry reductions and the similarity solutions of the equation. Based on the discrete analysis, they obtained two groups of discrete symmetry which lead to exact solutions of the Inviscid Burgers equation. Gangwei and Tianzhou [1] performed the Lie group analysis for the nonlinear perturbed Burgers equation and the time fractional nonlinear perturbed Burgers equation. They constructed all the point symmetries and the vector fields, subsequently investigated the symmetry reductions and finally obtained some exact explicit solutions of the equations. Burgers equation was also investigated by Okoya [6]. He determined all the Lie groups admitted by the equation and then used the symmetry transformation to establish all the global solutions corresponding to each Lie group admitted by the equation.

In this paper, we will investigate the vector fields, Lie groups, similarity reductions and the symmetry solutions to the modified type of Burgers equation:

$$u_t - 3u^2 u_x + u_{xx} = 0 \qquad (1)$$

using Lie symmetry analysis approach.

Here $u = u(x,t)$ represents the unknown function, x is the spatial coordinate in the propagation direction and t is the temporal coordinates, which occur in different context in mathematical physics [9], [10].

II. LIE SYMMETRY AND THE GEOMETRIC VECTOR FIELDS

A) *Geometric vector field of Eq. (1)*

Let $\Delta = u_t - 3u^2 u_x + u_{xx} = 0$ and the infinitesimal generator V of Eq. (1) to be of the form

$$V = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u} \qquad (2)$$

where the coefficients $\xi(x,t,u)$, $\tau(x,t,u)$ and $\eta(x,t,u)$ are to be determined.

For the symmetry condition to be satisfied by Eq. (1), then:

$$V^{(2)}\Delta\big|_{\Delta=0} = 0 \qquad (3)$$

where $V^{(2)}$ is the second prolongation of Eq.(2) given by:

$$V^{(2)} = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u} + \eta^x\frac{\partial}{\partial u_x} + \eta^t\frac{\partial}{\partial u_t} + \eta^{xx}\frac{\partial}{\partial u_{xx}} + \eta^{xt}\frac{\partial}{\partial u_{xt}} + \eta^{tt}\frac{\partial}{\partial u_{tt}} \qquad (4)$$

Substituting Eq. (4) into Eq. (3), the infinitesimal condition reduces to

$$\eta^t - 6uu_x\eta - 3u^2 n^x + \eta^{xx} = 0 \qquad (5)$$

which must be satisfied whenever $\Delta = 0$.

Substituting for η^t, η^x and η^{xx}, we have

$$\eta_t - \xi_x u_x + (\eta_u - \tau_t)u_t - \xi_u u_x u_t - \tau_u u_t^2 - 6uu_x\eta$$
$$-3u^2\left\{\eta_x + (n_u - \xi_x)u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t\right\}$$
$$+\eta_{xx} + (2\eta_{xu} - \xi_{xx})u_x - \tau_{xx}u_t + (n_{uu} - 2\xi_{xu})u_x^2$$
$$-2\tau_{xu}u_x u_t - \xi_{uu}u_x^3 - \tau_{uu}u_x^2 u_t + (\eta_u - 2\xi_x)u_{xx}$$
$$-2\tau_x u_{xt} - 3\xi_u u_x u_{xx} - \tau_u u_t u_{xx} - 2\tau_u u_x u_{xt} = 0$$

On replacing u_{xx} by $3u^2u_x - u_t$ wherever it occurs and equating the coefficients of various monomials in the first and second derivatives of u, we obtain the following determining equations.

$$\tau_u = \tau_x = 0 \quad (6)$$
$$\tau_u u^2 + 2\xi_u - 2\tau_{xu} = 0 \quad (7)$$
$$-\tau_t + 2u^2\tau_x - \tau_{xx} + 2\xi_x = 0 \quad (8)$$
$$-3u^2\xi_u + \eta_{uu} - 2\xi_{xu} = 0 \quad (9)$$
$$-\xi_t - 6u\eta + 2\eta_{xu} - \xi_{xx} - 3u^2\xi_x = 0 \quad (10)$$
$$\eta_t - 3u^2\eta_x + \eta_{xx} = 0 \quad (11)$$

We begin by solving eq. (6) to obtain

$$\tau = A(t)$$

where A is an arbitrary function.

From Eq. (7), we see that ξ is a function of x and t and thus the general solution of Eq.(8) is

$$\xi = \frac{1}{2}A_t x + B(t)$$

and Eq.(9) yields

$$\eta = C(x,t)u + D(x,t)$$

for some functions B, C and D. Substituting these results in (10) and (11), we obtain

$$-\frac{1}{2}A_{tt}x - B_t - 2C_x - 6Du - \left(6C + \frac{3}{2}A_t\right)u^2 = 0 \quad (12)$$
$$(D_t + D_{xx}) + (C_t + C_{xx})u - 3D_x u^2 - 3C_x u^3 = 0 \quad (13)$$

The functions A, B, C and D are independent of u. Therefore eq. (12) and (13) can be decomposed by equating the powers of u as follows:

$$D = 0 \quad (14)$$
$$C_x = 0 \quad (15)$$
$$C_t + C_{xx} = 0 \quad (16)$$
$$6C + \frac{3}{2}A_t = 0 \quad (17)$$
$$\frac{1}{2}A_{tt}x + B_t - 2C_x = 0 \quad (18)$$
$$D_x = 0 \quad (19)$$
$$D_t + D_{xx} = 0 \quad (20)$$

Using Eq. (14), (15), (16), (17) and (18) in turn we obtain

$$D = 0,\ C = k_1,\ A = -4k_1t + k_2, B = k_3 \quad (21)$$

where k_1, k_2 and k_3 are arbitrary constants.

Therefore the infinitesimals are given by

$$\xi = -2k_1x + k_3,\ \tau = -4k_1t + k_2,\ \eta = k_1u. \quad (22)$$

Substituting Eq. (22) into Eq. (2), we have

$$V = (-2k_1x + k_3)\frac{\partial}{\partial x} + (-4k_1t + k_2)\frac{\partial}{\partial t} + k_1u\frac{\partial}{\partial u} \quad (23)$$

From Eq. (23), we obtain the following vector fields:

$$V_1 = 2x\frac{\partial}{\partial x} + 4t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}$$
$$V_2 = \frac{\partial}{\partial t}$$
$$V_3 = \frac{\partial}{\partial x}$$

B) *Lie groups admitted by Eq. (1)*

The one parameter groups G_i admitted by Eq. (1) are determined by solving the corresponding Lie equations below:

$$V_1: \frac{dx^*}{d\varepsilon} = 2x^*,\ \frac{dt^*}{d\varepsilon} = 4t^*,\ \frac{du^*}{d\varepsilon} = -u^*$$
$$V_2: \frac{dx^*}{d\varepsilon} = 0,\ \frac{dt^*}{d\varepsilon} = 1,\ \frac{du^*}{d\varepsilon} = 0$$
$$V_3: \frac{dx^*}{d\varepsilon} = 1,\ \frac{dt^*}{d\varepsilon} = 0,\ \frac{du^*}{d\varepsilon} = 0$$

with the initial conditions that: $x^*_{\varepsilon=0} = 0$, $t^*_{\varepsilon=0} = t$ and $u^*_{\varepsilon=0} = 0$. This yields:

$$G_1: (x^*, t^*, u^*; \varepsilon) \to \left(xe^{2\varepsilon}, te^{4\varepsilon}, ue^{-\varepsilon}\right)$$
$$G_2: (x^*, t^*, u^*; \varepsilon) \to (x, t+\varepsilon, u)$$
$$G_3: (x^*, t^*, u^*; \varepsilon) \to (x+\varepsilon, t, u) \quad (24)$$

C) *The symmetry reduction of Eq. (1)*

Here we make use of the vector fields V_1, V_2, V_3 and $V_2 + \lambda V_3$ to reduce Eq. (1) to systems of ordinary differential equations (ODEs).

The symmetry reductions of Eq. (1) can be obtained by solving the characteristic equation given by:

$$\frac{dx}{\xi(x,t,u)}=\frac{dt}{\tau(x,t,u)}=\frac{du}{\eta(x,t,u)}$$

(a) For the vector field V_1, we have

$$\frac{dx}{2x}=\frac{dt}{4t}=\frac{du}{-u}$$

With the invariant function z taking the form $z=xt^{-\frac{1}{2}}$ and u expressed as

$$u=t^{-\frac{1}{4}}w(z) \qquad (25)$$

Differentiating Eq. (25) and substituting in Eq. (1), we obtain the second order ODE given by

$$-\frac{1}{4}w-\frac{1}{2}zw'-3w^2w'+w''=0 \qquad (26)$$

Where $w'=\frac{dw}{dz}$.

(a) For the vector field V_2, we have

$$\frac{dx}{0}=\frac{dt}{1}=\frac{du}{0}$$

The invariant function z takes the form $z=x$ while u is given by

$$u=w(z) \qquad (27)$$

Differentiating Eq. (27) and substituting in Eq. (1), we obtain

$$-3w^2w'+w''=0 \qquad (28)$$

where $w'=\frac{dw}{dz}$

(b) For the vector field V_3, we have

$$\frac{dx}{1}=\frac{dt}{0}=\frac{du}{0}$$

The invariant function $z=t$ satisfies

$$u=w(z) \qquad (29)$$

Differentiating Eq. (29) and substituting in Eq. (1), we obtain

$$w'=0 \qquad (30)$$

where $w'=\frac{dw}{dz}$.

(c) For the linear combination $V_2+\lambda V_3$, where $\lambda>0$ is the wave velocity, we have

$$\frac{dx}{\lambda}=\frac{dt}{1}=\frac{du}{0}$$

The invariant function z takes the form $z=x-\lambda t$ while u is given by

$$u=w(z) \qquad (31)$$

Differentiating Eq. (31) and substituting in Eq. (1), we arrive at:

$$-\lambda w'-3w^2w'+w''=0 \qquad (32)$$

where $w'=\frac{dw}{dz}$.

III. THE INVARIANT SOLUTIONS TO EQ. (1)

Here we consider the solutions of the reduced ordinary differential equations and then determine the invariant solutions to the original partial differential equation (1).

A) Exact power series solution of Eq. (26)

We seek a solution in power series of the form

$$w(z)=\sum_{n=o}^{\infty}c_nz^n \qquad (33)$$

Differentiating Eq. (33) and substituting in Eq. (26), we have

$$-\frac{1}{4}\sum_{n=0}^{\infty}c_nz^n-\frac{1}{2}\sum_{n=0}^{\infty}(n+1)c_{n+1}z^{n+1}$$
$$-3\left(\sum_{n=0}^{\infty}c_nz^n\right)\left(\sum_{n=0}^{\infty}c_nz^n\right)\left(\sum_{n=0}^{\infty}(n+1)c_{n+1}z^n\right)$$
$$+\sum_{n=0}^{\infty}(n+1)(n+2)c_{n+2}z^n=0$$

On relabeling, we obtain

$$-\frac{1}{4}c_o-\frac{1}{4}\sum_{n=1}^{\infty}c_nz^n-\frac{1}{2}\sum_{n=1}^{\infty}nc_nz^n$$
$$-3c_0^2c_1-3\sum_{n=1}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}(n+1-k)c_ic_{k-i}c_{n+1-k}z^n$$
$$+2c_2+\sum_{n=1}^{\infty}(n+1)(n+2)c_{n+2}z^n=0$$

Collecting the terms with similar powers of z together, we have

$$-\frac{1}{4}c_o-3c_o^2c_1+2c_2+\sum_{n=1}^{\infty}\Bigg(-\frac{1}{4}c_n-\frac{1}{2}nc_n$$
$$-3\sum_{k=0}^{n}\sum_{i=0}^{k}(n+1-k)c_ic_{k-i}c_{n+1-k}+(n+1)(n+2)c_{n+2}\Bigg)z^n=0$$
$$(34)$$

From Eq. (34), we have that for $n=0$,

$$c_2=\frac{c_0}{8}+\frac{3c_0^2c_1}{2} \qquad (35)$$

For $n \geq 1$, we obtain the recurrence relation

$$c_{n+2} = \frac{1}{(n+1)(n+2)}\left(\frac{c_n}{4} + \frac{nc_n}{2} + 3\sum_{k=0}^{n}\sum_{i=0}^{k}(n+1-k)c_i c_{k-i} c_{n+1-k}\right) \quad (36)$$

Where $c_i (i = 1, 2, ...)$ are arbitrary constants

For $n = 1, 2$, we obtain

$$c_3 = \frac{1}{6}\left(6c_0^2 c_2 + 6c_0 c_1^2 + \frac{3}{4}c_1\right)$$

$$c_4 = \frac{1}{12}\left(9c_0^2 c_3 + 18c_0 c_1 c_2 + 3c_1^3 + \frac{5}{4}c_2\right)$$

Hence the power series solution of Eq. (26) can be expressed as:

$$w(z) = c_0 + c_1 z + c_2 z^2 + \sum_{n=1}^{\infty} c_{n+2} z^{n+2}$$
$$= c_0 + c_1 z + \left(\frac{c_0}{8} + \frac{3c_o^2 c_1}{2}\right)z^2 + \sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}\left(\frac{c_n}{4} + \frac{nc_n}{2} + 3\sum_{k=0}^{n}\sum_{i}^{k}(n+1-k)c_i c_{k-i} c_{n+1-k}\right)z^{n+2}$$

There exact invariant power series solution of Eq. (1) is

$$u(x,t) = c_0 t^{-\frac{1}{4}} + c_1 x t^{-\frac{3}{4}} + \left(\frac{c_0}{8} + \frac{3c_o^2 c_1}{2}\right)x^2 t^{-\frac{5}{4}} + \sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}\left(\frac{c_n}{4} + \frac{nc_n}{2} + 3\sum_{k=0}^{n}\sum_{i}^{k}(n+1-k)c_i c_{k-i} c_{n+1-k}\right)x^{n+2} t^{-\frac{5+2n}{4}} \quad (37)$$

In mathematical and physical applications, the solution to Eq. (1) can conveniently be expressed in approximate form as

$$u(x,t) = c_0 t^{-\frac{1}{4}} + c_1 x t^{-\frac{3}{4}} + \left(\frac{c_0}{8} + \frac{3c_o^2 c_1}{2}\right)x^2 t^{-\frac{5}{4}} + \frac{1}{6}\left(6c_0^2 c_2 + 6c_0 c_1^2 + \frac{3}{4}c_1\right)x^3 t^{-\frac{7}{4}} + \frac{1}{12}\left(9c_0^2 c_3 + 18c_0 c_1 c_2 + 3c_1^3 + \frac{5}{4}c_2\right)x^4 t^{-\frac{9}{4}} + ... \quad (38)$$

B) The stationary solution to Eq. (28)

Integrating Eq.(28) and keeping the integration constant zero, we have:

$$w' - w^3 = 0 \quad (39)$$

Solving Eq.(39) we obtain

$$\frac{dw}{w^3} = dz$$

$$-\frac{1}{2w^2} = z + c$$

$$w^2 = -\frac{1}{2(z+c)}$$

$$w = \frac{i}{\sqrt{2(z+c)}}$$

Hence the exact analytic solution to Eq.(1) is given by :

$$u(x,t) = \frac{i}{\sqrt{2(x+c)}} = \frac{i}{\sqrt{2x+c}} \quad (40)$$

C) Solution to Eq. (30)

Integrating Eq.(30), we obtain $w = c$ where c is a constant of integration.

Thus we obtain a trivial solution to Eq.(1) given by

$$u(x,t) = c \quad (41)$$

D) Travelling wave solution to Eq. (32)

Integrating Eq.(32) and keeping the integration constant zero, we have

$$w' - \lambda w - w^3 = 0$$
$$\Rightarrow \frac{dw}{dz} = w(w^2 + \lambda) \quad (42)$$

On rearranging Eq.(42) and by partial fractions, we get

$$\frac{1}{\lambda}\left(\frac{dw}{w} - \frac{wdw}{w^2+\lambda}\right) = dz$$

On integration, we obtain

$$\ln w - \frac{1}{2}\ln(w^2+\lambda) = \lambda z + c$$

$$\Rightarrow \ln\left(\frac{w}{\sqrt{w^2+\lambda}}\right) = \lambda z + c$$

$$\Rightarrow w = \frac{c\sqrt{\lambda}e^{\lambda z}}{\sqrt{1 - c^2 \lambda e^{2\lambda z}}}$$

Where c is the constant of integration.

Thus the exact travelling wave solution to Eq.(1) is

$$u(x,t)=\frac{c\sqrt{\lambda}e^{\lambda(x-\lambda t)}}{\sqrt{1-c^2\lambda e^{2\lambda(x-\lambda t)}}} \quad (43)$$

IV. THE SYMMETRY SOLUTIONS TO EQ.(1)

According to Olver P.J [7], if $u=w(x,t)$ is a solution of Eq.(1), so are the functions:

$$G_1(\varepsilon)w(x,t)=w(xe^{-2\varepsilon},te^{-4\varepsilon})e^{-\varepsilon}$$

$$G_2(\varepsilon)w(x,t)=w(x,t-\varepsilon)$$

$$G_3(\varepsilon)w(x,t)=w(x-\varepsilon,t)$$

That is, a symmetry group of Eq.(1) is a local group of transformation G with the property that whenever $u=w(x)$ is a solution of Eq.(1) and whenever $g.w$ is defined for $g\in G$, then $u=g.w$ is also a solution of Eq.(1).

Thus using G_1, the symmetry solutions of Eq.(1) can be given by expressing equations (37), (40), (41) and (43) as follows:

$$u^*(x^*,t^*)=\Bigg\{c_0(t^*e^{-4\varepsilon})^{-\frac{1}{4}}+c_1(x^*e^{-2\varepsilon})(t^*e^{-4\varepsilon})^{-\frac{3}{4}}$$
$$+\left(\frac{c_0}{8}+\frac{3c_o^2c_1}{2}\right)(x^*e^{-2\varepsilon})^2(t^*e^{-4\varepsilon})^{-\frac{5}{4}}$$
$$+\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}\left(\frac{c_n}{4}+\frac{nc_n}{2}\right.$$
$$\left.+3\sum_{k=0}^{n}\sum_{i}^{k}(n+1-k)c_ic_{k-i}c_{n+1-k}\right)(x^*e^{-2\varepsilon})^{n+2}(t^*e^{-4\varepsilon})^{-\frac{5+2n}{4}}\Bigg\}e^{-\varepsilon} \quad (44)$$

$$u^*(x^*,t^*)=\frac{i}{\sqrt{2x^*e^{-2\varepsilon}+c}}e^{-\varepsilon} \quad (45)$$

$$u^*(x^*,t^*)=ce^{-\varepsilon} \quad (46)$$

$$u^*(x^*,t^*)=\frac{c\sqrt{\lambda}e^{\lambda(x^*e^{-2\varepsilon}-\lambda t^*e^{-4\varepsilon})}}{\sqrt{1-c^2\lambda e^{2\lambda(x^*e^{-2\varepsilon}-\lambda t^*e^{-4\varepsilon})}}}e^{-\varepsilon} \quad (47)$$

G_2 and G_3 can be used in a similar manner.

V. CONCLUSION

In this paper, we have obtained the geometric vector fields, Lie groups and the symmetry reduction of the modified type of Burgers equation (1) using Lie symmetry analysis method. Moreover, all the group invariant solutions to the equation have been considered and the exact symmetry solutions to the equation determined by transforming a known invariant solution by all the possible group elements.

REFERENCES

[1] **Gangwei W, Tianzhou X (2015),** Invariant analysis and explicit solutions of the time fractional nonlinear perturbed Burgers equation, *Nonlinear analysis: modelling and control*, Vol. 20, No. 4, 570-584

[2] **Hanze l, Jibin L, Quanxin Z (2008**), Lie symmetry analysis and the exact explicit solutions to the general Burgers equation, *J. Math. Anal.* 228, 1-9.

[3] **Hanze L, Jibin L, Lei L (2010),** Lie symmetry analysis, optimal systems and the exact solutions to the fifth –order KdV types of equations, *J.Math. Anal. App 368,* 551-558.

[4] **Hongwei Y, Yunglong S, Baoshu Y, Huanhe D (2012),** Discrete symmetries analysis and exact solutions of the Inviscid Burgers equation, *Hindawi publishing corporation*, ID 908975

[5] **Motsepa T (2016),** Symmetry analysis, conservation laws and exact solutions of certain nonlinear partial differential equation, *PhD thesis, Mafikeng Campus of North-west India*.

[6] **Oduor Okoya M.E, Ongati N.O, Aminer T.J.O, Nyakinda J.O (2013),** Symmetry group approach to the solution of generalized Burgers equation, *Applied mathematical Sciences*, Vol. 7, No. 15, 717-725.

[7] **Olver P.J (1986),** Application of Lie groups to differential equations, *Springer-Verlag, New York , NY, USA.*

[8] **Spirana N, Chatanin W (2016),** Lie symmetry analysis and exact solutions to the Quintin nonlinear Beam equation, *Malaysian Journal of mathematical Sciences,* 10(1), 61-68.

[9] **Zhang Y (2016),** Lie symmetry analysis and the exact solutions of the Sharma-Tasso-Olever equation, *International Journal of Applied Mathematics*, 46:2

[10] **Zhang Y (2017),** Lie symmetry analysis and the exact solutions of the Sawada-Kotera equation, *Turk J Math* 41, 158-167.