Some aspects of derivation in ternary semirings.

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Abstract

In this paper we are concerned with the study of some properties of ideals in ternary semirings. The notion of idempotent, invertible and strong invertible elements has been introduced. We also give another notion of commutativity and introduce the notion of cyclic commutativity. We construct some prime ideals in connection with the inverse element.

The second part of the paper is devoted to the study of derivations on particular elements of such algebraic structures. We also give some examples of co-ideals.

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1 Introduction and preliminary Notes

D.H. Lehmer [17] have started the concept of ternary algebraic system in 1932. After that several authors have extended the results in many ways. Dutta and Kar [3] have initiated the notion of ternary semirings which is generalization of ternary ring introduced by W. G. Lister [18] in 1971. Good and Hughes [14] introduced the notion of bi-ideals in semigroups. S. Kar [15] generalized the concepts of quasi ideals and bi-ideals in ternary semirings and give many characterizations in terms of the same. In this paper we proceed with the study some other simple and useful properties of ideals and the derivation on ternary semirings. This study will be an introduction for a forcomming study on the relationship between derivation and Jordan derivation on some particular ternary semirings.

Definition 1.1 *A non empty set S together with a binary operation called addition and ternary operation called multiplication, denoted by juxtaposition is said to be a ternary semiring if S is an additive commutative semigroup satisfying the following conditions:*

1. $(abc)de = a(bcd)e = ab(cde)$,
2. $(a+b)cd = acd + bcd$,
3. $a(b+c)d = abd + acd$,
4. $ab(c+d) = abc + abd, forall a, b, c, d, e \in S$.

Remark 1.2 *Any binary operation $\cdot$ on a set S defines a ternary operation θ by*

$$\theta(a, b, c) = (a \cdot b) \cdot c$$

Definition 1.3 *Let S be a ternary semiring. If there exists an element $0 \in S$ such that $0 + x = x = x + 0$ and $0xy = x0y = xy0 = 0$ for all $x, y \in S$. If such element exists it is unique and then call it the zero element or simply the zero of the ternary semiring S. In this case we say that S is a ternary semiring with zero.*

Throughout this paper, S will always denote a ternary semiring with zero and unless, otherwise stated a ternary semiring means a ternary semiring with zero.

LetA, B, C be three subsets of S. Then by ABC, we mean the set of all finite sums of the form

$$\sum a_i b_i c_i \quad \text{with} \quad a_i \in A, b_i \in B, c_i \in C.$$

Definition 1.4 *An additive subsemigroup H of S is called a ternary subsemiring if*

$$a \cdot b \cdot c \in H; \quad \forall a, b, c \in H \quad \text{and} \quad 0 \in H.$$

2 Ideals of ternary semirings

Definition 2.1 . *An additive subsemigroup I of S is called a left (right, bilateral) ideal of S if $a.b.i$ (respectively $i.a.b, a.i.b$) $\in I$ for all $a; b \in S$ and $i \in I$. An ideal is an ideal which is left, right and bilateral ideal.*

Remark 2.2 *If a ternary semiring S has a zero, then for any left (resp. right, lateral) ideal I; 0 is also in I.*

Definition 2.3 *An ideal I of a ternary semiring S is called left cancelable if:*

$$a, b, abc \in I \implies c \in I$$

Definition 2.4 *An ideal I of a ternary semiring S is called right cancelable if:*

$$b, c, abc \in I \implies a \in I$$

Example 2.5 *We can view the set $\mathbb{N}$ endowed with $+$ and $\times$ as a ternary semiring with zero. The subset of even integers is an ideal.*

Definition 2.6 *An element a in a ternary semiring S is called regular if for all $x, y \in S$ the equality $axa = aya$ implies $x = y$. A ternary semiring S is called regular if all of its elements excepts 0 are regular.*

Example 2.7 *The ternary semiring $\mathbb{Z}^-$ is regular.*

Example 2.8 *In the ternary semiring $S = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \ / \ a, b, c \in \mathbb{Z}^-\}$; $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ is regular if and only if $a \neq 0$ and $c \neq 0$. S is then not regular(Hint: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is not regular).*

Proposition 2.9 *Let S be a ternary semiring; the set $I_a = \{xay, \ / \ x, y \in S\}$ is a left and a right ideal of S.*

Proof 2.10 *Let $xay \in I_a$ and $\alpha, \beta \in S$; then $\alpha\beta(xay) = (\alpha\beta x)ay \in I_a$ using the property 1 of definition 1.1. In the other hand using the same property we can write: $(xay)\alpha\beta = xa(\alpha\beta y)$. The factor $(\alpha\beta y)$ is in S so $(xay)\alpha\beta \in I_a$ and I_a is also a right ideal.*

Definition 2.11 *An element a of a ternary semiring S is called an idempotent element if $a^3 = a$. A ternary semiring S is called an idempotent ternary semiring if every element of S is an idempotent.*

Example 2.12 *The only idempotent of the ternary semiring of example 2.8 is the matrix: $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.*

Definition 2.13 *A ternary semiring S is said to be*

1. *commutative if $abc = acb = cba = bac \ \ \forall a, b, c \in S$.*
2. *cyclicly commutative if $abc = bca = cab \ \ \forall a, b, c \in S$.*

Remark 2.14 *If S is commutative, then S is cyclicly commutative. The converse is false.*

Definition 2.15 *Let S be a ternary semiring, the center of S denoted by $\mathcal{Z}(S)$ is the set defined by:*

$$\mathcal{Z}(S) = \{x \in S \ \ / \ \ xyz = yzx) \ \ \textit{for all} \ \ y, z \in S\}.$$

Proposition 2.16 *S is cyclicly commutative if and only if $\mathcal{Z}(S) = S$.*

Proof 2.17 *Let $x \in S$ for $y, z \in S$ we have by the cyclic commutativity $xyz = yzx$.*

The converse: $x.y.z = y.z.x = z.x.y$ where we consider for the first equality x as an element of $\mathcal{Z}(S)$ and for the second y as an element of $\mathcal{Z}(S)$.

Definition 2.18 *An element 1 of a ternary semiring S is called unity of S if*

$$\forall x, y \in S; \quad 1xy = x1y = xy1.$$

and

$$\forall x \in S; \quad x = 11x = x11 = 1x1.$$

Proposition 2.19 *Let S be a ternary semiring with unity and such $\forall x, y, z \in S; xyz = (xy1)z1$ then $\forall a \in S$ the set I_a is an ideal of S.*

Proof 2.20 *Let $xay \in I_a$ and $\alpha, \beta \in S$; then $\alpha(xay)\beta = (\alpha xa)y\beta = ((\alpha x1)a1)y\beta = (\alpha x1)(a1y)\beta = (\alpha x1)a(1y\beta) \in I_a$ since $(\alpha x1)$ and $(1y\beta)$ are elements of S. Finally I_a is a bilateral ideal and by proposition 1.8 it is an ideal.*

Definition 2.21 *Let S be a ternary semiring with unity 1. An element a of S is said to be invertible if there exists $b \in S$ such:*

$$aba = a \ \ \textit{and} \ \ bab = b.$$

Proposition 2.22 *Let S be a ternary semiring with unity. If b is an inverse of a then aba is an inverse of bab.*

Proof 2.23 *$aba = a$ and $bab = b$ imply that $(bab)(aba)(bab) = bab$ and $(aba)(bab)(aba) = aba$ and then aba is an inverse of bab.*

Definition 2.24 *An ideal P of a ternary semiring S is called a prime ideal if $\forall x; y; z \in S; xyz \in P \implies x \in P$ or $y \in P$ or $z \in P$.*

Proposition 2.25 *Let S be a commutative or cyclicly commutative ternary semiring with unity. If a is invertible and its inverse is b; then:*

1. *a is always in the ideal I_a and then I_a is prime,*
2. $I_a = I_b$

Proof 2.26 1. *a is invertible and b is an inverse of a, then $a = aba = baa \in I_a$. If $xay \in I_a$, we always have $a \in I_a$, so I_a is prime.*

2. *Let $xay \in I_a$. We have $a = aba = baa$ so $xay = x(baa)y = xb(aay) \in I_b$ and finally $I_a \subset I_b$. We proceed in the same manner to prove that $I_b \subset I_a$.*

Proposition 2.27 *Let S be a ternary semiring with unity. If an invertible element a is regular; then its inverse is unique.*

Proof 2.28 *Let $b; c$ be two inverses of a then $aba = a = aca$ which implies that $b = c$ using the regularity of a.*

Definition 2.29 *An $AG-$ternary semiring S is a ternary semiring such $abc = cba$, $\forall a, b, c \in S$. It is obvious that a commutative ternary semiring is an $AG-$ternary semiring*

Remark 2.30 *In an $AG-$ternary semiring S; I is a left ideal $\Longleftrightarrow$ I is a right ideal.*

Definition 2.31 *An element a of a ternary semiring S is called strongly invertible is there exists an element b in S such:*

$$abx = xab = bax = xba = x; \quad \forall x \in S.$$

Proposition 2.32 *Any strongly invertible element of a ternary semiring is invertible.*

Proof 2.33 *a is strongly invertible then there exists $b \in S$ such in particular $abx = x$ and $bax = x$; $\forall x \in S$. So in the first case we take $x = a$ and in the second case the relation is also true for $x = b$ and then a is invertible.*

Proposition 2.34 *Let a be an invertible element and b an inverse of a. If the mapping:*

$$\begin{array}{llll} f: & S & \longrightarrow & S \\ & x & \longmapsto & abx \end{array}$$

is onto then a is strongly invertible.

Proof 2.35 *Let $y \in S$, we will prove in the first step that $aby = y$. f is onto then there is $x \in S$ such $abx = y$, so using that $aba = a$ we get $(aba)bx = abx \iff ab(abx) = abx \iff aby = y$*

3 Derivation on ternary semirings

Definition 3.1 *Let S, S' be two ternary semirings, a mapping $f : S \longrightarrow S'$ is a ternary semiring morphism if* $\forall x, y, z \in S$;

1. $f(x + y) = f(x) + f(y)$,
2. $f(x.y.z) = f(x).f(y).f(z)$.

Proposition 3.2 *If 1 is a unity of the ternary semiring S and f is onto, then $f(1)$ is a unity of S'.*

Proof 3.3 *The proof is trivial.*

Proposition 3.4 *Let $f : S \longrightarrow S'$ be a ternary semirings morphism where S has a zero element.*

1. *The kernel $ker f = \{x \in S \ / \ f(x) = 0\}$ is an ideal of S.*
2. *If f is onto then $f(0) = 0$.*

Proof 3.5

1. *If $x, y \in ker f$ then $f(x + y) = f(x) + f(y) = 0 + 0 = 0$; so $ker f$ is an additive subsemigroup.*

   Let $x \in ker f, y, z \in S$, then $f(xyz) = f(x)f(y)f(z) = 0f(y)f(z) = 0$ so $xyz \in ker f$ and $ker f$ is a right ideal. The same arguments can be used to prove that $ker f$ is both a left and a lateral ideal. In conclusion $ker f$ is an ideal of S.

2. *$\forall y, z \in S'$; $\exists x, a \in S$ such that $f(x) = y, f(a) = z$. So $y + f(0) = f(x) + f(0) = f(x + 0) = f(x) = y = f(0) + y$ and*

   $f(0)yz = f(0xa) = f(0), f(x)f(0)f(a) = f(x0a) = f(0)$ and $f(x)f(a)f(0) = f(xa0) = f(0)$ so $f(0)$ is a zero of S' and by the unicity $f(0) = 0$.

Definition 3.6 *An element a of a ternary semiring S is an idempotent (resp. pseudo-idempotent) if: $aaa = a$ (resp.$1aa = 1$). Where 1 is a unity of S for the multiplication.*

Proposition 3.7 *Any idempotent is invertible and has itself as an inverse.*

Definition 3.8 *As for the rings, the characteristic of a ternary semiring S is the least non null integer k if it exists such $kx = 0 \ \ \forall x \in S$, otherwise the characteristic will be 0*

Proposition 3.9 *If $e \in S$ is a idempotent where S is of characteristic 3, then $1 + e$ is also idempotent.*

Proof 3.10 $(1+e)(1+e)(1+e) = 1 + 3e + 3(ee1) + eee = 1 + eee = 1 + e$ *so* $1 + e$ *is an idempotent.*

Proposition 3.11 *If* a *is a regular pseudo-idempotent element of a ternary semiring* S *then the mapping:*

$$\begin{array}{rccc} f_a: & S & \longrightarrow & S \\ & x & \longmapsto & axa \end{array}$$

is an isomorphism.

Proof 3.12 *Let* $x, y, z \in S$

1. $f_a(x+y) = a(x+y)a = axa + aya = f_a(x) + f_a(y)$.
2. $f_a(xyz) = a(xyz)a$ *and we can write:*

   $xyz = (x11)y(11z) = (x(aa1)1)y(1(11a)z) = (xa(a11))y((11a)az) = (xaa)y(aaz)$

   and then

   $a(xyz)a = a[(xaa)y(aaz)]a = a(xaa)[y(aaz)a] = a(xaa)[ya(aza)] = (axa)[a[ya(aza)]] = (axa)(aya)(aza) = f_a(x)f_a(y)f_a(z)$.
3. *The regularity implies that* f_a *is one-to-one.*
4. *Let now* $y \in S$ *the element* $x = aya \in S$ *and verifies* $f_a(x) = a(aya)a = a(1(aya)1)a = (a1(aya))1a = ((a1a)ya)1a = (1ya)1a = 1y(a1a) = 1y1 = y$ *and* f_a *is onto.*

Definition 3.13 *Let* S *be a ternary semiring. An additive mapping* $d : S \longrightarrow S$ *is called*

1. *a derivation if:*

$$d(xyz) = d(x)yz + xd(y)z + xyd(z);\ \forall x, y, z \in S$$

2. *a Jordan derivation if:*

$$d(xxx) = d(x)xx + xd(x)x + xxd(x);\ \forall x \in S$$

Remark 3.14 *It is obvious that a derivation is a Jordan derivation.*

Proposition 3.15 *Let* D *be a Jordan derivation on an additively regular ternary semiring* S*, then:* $\forall a, b \in S$;

$$D(a^2b+aba+ab^2+ba^2+bab+b^2a) = D(a)ab+aD(a)b+a^2D(b)+D(a)ba+aD(b)a+$$
$$+abD(a)+D(a)b^2+aD(b)b+abD(b)+D(b)a^2+bD(a)a+baD(a)+D(b)ab+bD(a)b+$$
$$+baD(b) + D(b)ba + bD(b)a + b^2D(a).$$

Proof 3.16 *By linearizing and by using the additive regularity of* S *and from the following relation;*

$$D[(a+b)^3] = D(a+b)(a+b)(a+b)+(a+b)D(a+b)(a+b)+(a+b)(a+b)D(a+b) =$$

$$= (D(a)+D(b))(a+b)(a+b)+(a+b)(D(a)+D(b))(a+b)+(a+b)(a+b)(D(a)+D(b)).$$

we get:

$$D(a^2b+aba+ab^2+ba^2+bab+b^2a) = D(a)ab+aD(a)b+a^2D(b)+D(a)ba+aD(b)a+$$

$$+abD(a)+D(a)b^2+aD(b)b+abD(b)+D(b)a^2+bD(a)a+baD(a)+D(b)ab+bD(a)b+$$

$$+baD(b) + D(b)ba + bD(b)a + b^2D(a).$$

Proposition 3.17 *If a unitary ternary additively regular semiring* S *is of characteristic* $\neq 2$ *and* d *is a derivation on* S*, then* $d(1) = 0$.

Proof 3.18 *We have* $111 = 1$ *so* $d(111) = d(1) \iff d(1)11 + 1d(1)1 + 11d(1) = d(1) \iff 3d(1) = d(1)$. *Using the additive regularity we deduce that* $2d(1) = 0$ *and then by the condition on the characteristic we get* $d(1) = 0$.

Proposition 3.19 *If* $e \in S$ *is a idempotent where* S *is additively regular of characteristic* $\neq 3$ *and* d *is a derivation on* S*, then* $d(e) + 1d(e)e + 1ed(e) = 0$.

Proof 3.20 *If* e *is an idempotent then by the proposition 3.9;* $1 + e$ *is an idempotent. So from* $d(1) = 0$ *we have:*

$d((1+e)(1+e)(1+e)) = d(1+e)(1+e)(1+e)+(1+e)d(1+e)(1+e)+(1+e)(1+e)d(1+e) = (d(e))(11+1e+e1+ee)+(1+e)d(e)(1+e)+(11+1e+e1+ee)d(1) = 4d(e) + d(1ee) + d(e1e) + d(ee1) = 4d(e) + 3(1d(e)e + 1ed(e))$.

In the other hand $d((1 + e)(1 + e)(1 + e)) = d(1 + e) = d(e)$*; so*
$d(e) + 1d(e)e + 1ed(e) = 0$.

Definition 3.21 *Let* S *be a ternary semiring. An additive mapping* $D : S \longrightarrow S$ *is called a generalized derivation if there exists a derivation* d *of* S *such:*

$$D(xyz) = D(x)yz + xD(y)z + xyd(z);\ \forall x, y, z \in S$$

Example 3.22 .

1. *If* $S = Z^-[X]$ *the set of polynomials on the indeterminate* X *endowed with the polynomials addition and multiplication. This set is a ternary semiring and the usual polynomial derivation is a derivation on* S.

2. *Let* $S = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \ \ / \ \ a, b, c \in \mathbb{Z}^-\}$. *If* $d(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}) = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ *then* d *is evidently a derivation on the ternary semiring* $(S, +, \times)$.

Proposition 3.23 *Let* d *be a derivation on a ternary semiring* S. *And suppose that* $a \in S$ *is such* $d(a) = a$ *then the ideal* I_a *is stable by* d.

Proof 3.24 *Let* $xay \in I_a$ *then* $d(xay) = d(x)ay + xd(a)y + xad(y) = d(x)ay + xay + xad(y) \in I_a$ *as the sum of three terms of* I_a.

Proposition 3.25 *Let* D *be a Jordan derivation on a ternary semiring* S. *The ideal* I_a *is partially stable by* D *(i.e.* $\forall x \in I_a;\ D(xxx) \in I_a$*).*

Proof 3.26 *Let* $xay \in I_a$ *then* $D((xay)(xay)(xay)) = D(xay)(xay)(xay) + (xay)D(xay)(xay) + (xay)(xay)D(xay)$. *the element* (xay) *is in* I_a *and* I_a *is a left and a right ideal then any of the three terms in the previous sum is in* I_a *so is* $D((xay)(xay)(xay))$.

Theorem 3.1 *Let* d *be a derivation on a ternary semiring* S, $a \in \mathcal{Z}(S)$ *invertible and* b *is one of its inverses. Then*

$$3[d(a)d(b)d(a)]ab + 3[d(a)d(b)d(b)]aa + 3[d(a)d(b)b]ad(a) + 2aa[d(a)d(a)d(b)] +$$

$$ab[d(a)d(a)d(b)] + 4aa[d(b)d(b)d(a)] + [d(b)d(b)d(a)]ab + 2a[d(a)bd(a)]d(a) +$$

$$a[d(a)bd(a)]bd(a) + aa[d(b)d(a)d(b)] + 2a(aaa)[d(b)d(b)d(b)] +$$

$$2ab[(d(a)bd(a))bd(a)] + a[d(a)d(b)b + d(b)d(a)b]d(a) = d(a)d(b)d(a)$$

Lemma 3.27 *Let* a, b *as in the previous theorem. Then*

1. $$[d(a)ba][d(b)ab][d(a)ba] = a[d(a)d(b)d(a)]b.$$

2. $$[d(a)ba][d(b)ab][ad(b)a] = a[d(a)d(b)d(b)]a.$$

3. $$[d(a)ba][d(b)ab][abd(a)] = a[d(a)d(b)b]d(a).$$

Proof 3.28 *1.*

$$[d(a)ba][d(b)ab][d(a)ba] = d(a)[ba(d(b)ab)][d(a)ba] = d(a)[ab(ad(b)b)][d(a)ba] =$$
$$d(a)[(aba)d(b)b)][d(a)ba] = d(a)[(ad(b)b)][d(a)ba] = d(a)[d(b)ab][d(a)ba] =$$
$$d(a)d(b)[ab(d(a)ba)] = d(a)d(b)[(aba)d(a)b] = d(a)d(b)[ad(a)b] =$$
$$[d(a)d(b)a]d(a)b = [ad(a)d(b)]d(a)b = a[d(a)d(b)d(a)]b.$$

2.

$$[d(a)ba][d(b)ab][ad(b)a] = d(a)[ba(d(b)ab)][ad(b)a] = d(a)[ab(ad(b)b)][ad(b)a] =$$
$$d(a)[(aba)d(b)b)][ad(b)a] = d(a)[(ad(b)b)][ad(b)a] = d(a)[d(b)ab][ad(b)a] =$$
$$d(a)d(b)[ab(ad(b)a)] = d(a)d(b)[(aba)d(b)a] = d(a)d(b)[ad(b)a] =$$
$$[d(a)d(b)a]d(b)a = [ad(a)d(b)]d(b)a = a[d(a)d(b)d(b)]a.$$

3.

$$[d(a)ba][d(b)ab][abd(a)] = d(a)[ba(d(b)ab)][abd(a)] = d(a)[ab(ad(b)b)][abd(a)] =$$
$$d(a)[(aba)d(b)b)][abd(a)] = d(a)[(ad(b)b)][abd(a)] = d(a)[d(b)ab][abd(a)] =$$
$$d(a)d(b)[ab(abd(a))] = d(a)d(b)[(aba)bd(b)] = d(a)d(b)[abd(b)] =$$
$$[ad(a)d(b)]bd(a) = a[d(a)d(b)b]d(a).$$

Lemma 3.29 *Let a, b as in the previous theorem. Then*

1.
$$[d(a)ba][bd(a)b][d(a)ba] = a[d(a)bd(a)][bd(a)b].$$

2.
$$[d(a)ba][bd(a)b][ad(b)a] = aa[d(a)d(a)d(b)].$$

3.
$$[d(a)ba][bd(a)b][abd(a)] = a[d(a)d(b)b]d(a).$$

Proof 3.30 *1.*

$$[d(a)ba][bd(a)b][d(a)ba] = d(a)[ba(bd(a)b)][d(a)ba] = d(a)[ba(bd(a)b)][d(a)ba] =$$
$$d(a)[(bab)d(a)b)][d(a)ba] = d(a)[bd(a)b)][d(a)ba] = a[d(a)bd(a)][bd(a)b].$$

$$[d(a)ba][bd(a)b][ad(b)a] = d(a)[ba(bd(a)b)][ad(b)a] = d(a)[ba(bd(a)b)][ad(b)a] =$$
$$d(a)[(bab)d(a)b)][ad(b)a] = [d(a)bd(a)]b[d(b)aa] = d(a)[d(a)bd(b)]a.$$

2.

$$[d(a)ba][bd(a)b][abd(a)] = d(a)[ba(d(b)ab)][abd(a)] = d(a)[(bab)d(a)b)][abd(a)] =$$

$$d(a)[bd(a)b][abd(a)] = d(a)[bd(a)b]d(a).$$

Lemma 3.31 *Let a, b as in the previous theorem. Then*

1.

$$[d(a)ba][bad(b)][d(a)ba] = a[d(a)d(b)d(a)]b.$$

2.

$$[d(a)ba][bad(b)][ad(b)a] = a[d(a)d(b)d(b)]a.$$

3.

$$[d(a)ba][bad(b)][abd(a)] = a[d(a)d(b)b]d(a).$$

Proof 3.32 *1.*

$$[d(a)ba][bad(b)][d(a)ba] = d(a)[ba(bad(b)][d(a)ba] = d(a)[(aba)bd(b)][d(a)ba] =$$

$$d(a)[(aba)d(b)d(a)]b] = [d(a)d(b)d(a)]ab.$$

2.

$$[d(a)ba][bad(b)][ad(b)a] = d(a)[ba(bad(b)][ad(b)a] = d(a)[(aba)bd(b)][ad(b)a] =$$

$$d(a)[abd(b)][ad(b)a] = d(a)[(aba)d(b)d(b)]a = [d(a)d(b)d(b)]aa.$$

3.

$$[d(a)ba][bad(b)][abd(a)] = d(a)[ba(bad(b)][abd(a)] = d(a)[(aba)bd(b)][abd(a)] =$$

$$d(a)[abd(b)][abd(a)] = d(a)[ad(b)b]d(a) = [d(a)d(b)b]d(a)a.$$

Lemma 3.33 *Let a, b as in the previous theorem. Then*

1.

$$[ad(b)a][d(b)ab][d(a)ba] = a[d(b)d(b)d(a)]a.$$

2.

$$[ad(b)a][d(b)ab][ad(b)a] = (d(b)^3]a[a^3].$$

3.

$$[ad(b)a][d(b)ab][abd(a)] = a^2(d(b)d(b)d(a)).$$

Proof 3.34 *1.*

$$[ad(b)a][d(b)ab][d(a)ba] = d(b)[aa(d(b)ab)][d(a)ba] = d(b)[ad(b)(aba)][d(a)ba] =$$

$$d(b)[ad(b)a][d(a)ba] = ad(b)[d(b)a(d(a)ba)] = ad(b)[d(b)(a(d(a)b)a)] =$$

$$ad(b)[d(b)d(a)(aba)] = ad(b)[d(b)d(a)a] = a(d(b)d(b)d(a))a.$$

2.

$$[ad(b)a][d(b)ab][ad(b)a] = d(b)[aa(d(b)ab)][ad(b)a] = d(b)[ad(b)(aba)][ad(b)a] =$$

$$d(b)[(ad(b)a)][ad(b)a] = d(b)[ad(b)a][ad(b)a] = [d(b)ad(b)]a[ad(b)a] =$$

$$d(b)d(b)[ad(b)(a^3)] = d(b)d(b)(d(b)a(a^3)] = [d(b)]^3a[a^3].$$

3.

$$[ad(b)a][d(b)ab][abd(a)] = d(b)[aa(d(b)ab)][abd(a)] = d(b)[ad(b)(aba)][abd(a)] =$$

$$d(b)[(ad(b)a)][abd(a)] = d(b)[ad(b)a][abd(a)] = [d(b)ad(b)]a[abd(a)] =$$

$$(d(b)ad(b))ad(b) = d(b)d(b)(ad(a)a) = a^2(d(b)d(b)d(a)).$$

Lemma 3.35 *Let a, b as in the previous theorem. Then*

1.

$$[ad(b)a][bad(b)][d(a)ba] = a[d(b)d(b)d(a)]a.$$

2.

$$[ad(b)a][bad(b)][ad(b)a] = (d(b)^3]a[a^3].$$

3.

$$[ad(b)a][bad(b)][abd(a)] = (d(b)d(b)d(a))a^2.$$

Proof 3.36 *1.*

$$[ad(b)a][bad(b)][d(a)ba] = ad(b)[a(bad(b))(d(a)ba)] = ad(b)[(aba)d(b))(d(a)ba)] =$$

$$ad(b)[ad(b))(d(a)ba)] = ad(b)[d(b)(ad(a)b)a)] = ad(b)[d(b)(d(a)ab)a)] =$$

$$ad(b)[d(b)d(a)(aba)] = ad(b)[d(b)d(a)(aba)] = ad(b)[d(b)d(a)a] = a[d(b)d(b)d(a)]a.$$

2. Using the properties of the ternary semiring it is easy to get:

$$[ad(b)a][bad(b)][ad(b)a] = [d(b)]^3a(a^3).$$

3. *We can also prove that:*

$$[ad(b)a][bad(b)][abd(a)] = [d(b)d(b)d(a)]a^2.$$

Lemma 3.37 *1.*

$$[ad(b)a][bd(a)b][d(a)ba] = [d(b)d(a)d(a)]ab.$$

2.

$$[ad(b)a][bd(a)b][ad(b)a] = ([d(b)d(a)d(b)]aa.$$

3.

$$[ad(b)a][bd(a)b][abd(a)] = a[d(b)d(a)b]d(a).$$

Proof 3.38 *1.*

$$[ad(b)a][bd(a)b][d(a)ba] = d(b)[aa(bd(a)b)](d(a)ba) = d(b)[(aba)d(a)b](d(a)ba) =$$

$$d(b)[ad(a)b](d(a)ba) = d(b)d(a)[ab(d(a)ba)] = d(b)d(a)[(aba)d(a)b] = [d(b)d(a)d(a)]ab.$$

2. *It is easy to get the following equality:*

$$[ad(b)a][bd(a)b][ad(b)a] = [d(b)d(a)d(b)]aa.$$

3. *The following equality can be derived using the same calculus:*

$$[ad(b)a][bd(a)b][abd(a)] = a[d(b)d(a)b]d(a)].$$

With analogous calculus we can prove the following lemma.

Lemma 3.39 *Let a, b as in the previous theorem. Then*

1. $[abd(a)][d(b)ab][d(a)ba] = a[d(a)d(b)d(a)]b.$
2. $[abd(a)][d(b)ab][ad(b)a] = aa[d(a)d(b)d(b)].$
3. $[abd(a)][d(b)ab][abd(a)] = a[d(a)d(b)b]d(a).$
4. $[abd(a)][bd(a)b][d(a)ba] = a[bd(a)b][d(a)bd(a)].$
5. $[abd(a)][bd(a)b][ad(b)a] = ab[d(a)d(a)d(b)].$
6. $[abd(a)][bd(a)b][abd(a)] = ab[(d(a)bd(a))bd(a)].$
7. $[abd(a)][abd(a)][d(a)ba] = ad(a)[bd(a)d(a)].$
8. $[abd(a)][abd(a)][ad(b)a] = aa[d(a)d(a)d(b)].$

9. $[abd(a)][abd(a)][abd(a)] = ad(a)[bd(a)d(a)]$.

Proof 3.40 *(of the theorem)*

$3[d(a)d(b)d(a)]ab+3[d(a)d(b)d(b)]aa+3[d(a)d(b)b]ad(a)+2aa[d(a)d(a)d(b)]+$

$ab[d(a)d(a)d(b)]+4aa[d(b)d(b)d(a)]+[d(b)d(b)d(a)]ab+2a[d(a)bd(a)]d(a)+$

$a[d(a)bd(a)]bd(a)+aa[d(b)d(a)d(b)]+2a(aaa)[d(b)d(b)d(b)]+$

$2ab[(d(a)bd(a))bd(a)]+a[d(a)d(b)b+d(b)d(a)b]d(a)=$

$a[d(a)d(b)d(a)]b+a[d(a)d(b)d(b)]a+a[d(a)d(b)b]d(a)$

$+a[d(a)bd(a)][bd(a)b]+aa[d(a)d(a)d(b)]+a[d(a)d(b)b]d(a)$

$+a[d(a)d(b)d(a)]b+a[d(a)d(b)d(b)]a+a[d(a)d(b)b]d(a)$

$+a[d(b)d(b)d(a)]a+(d(b)^3]a[a^3]+a^2(d(b)d(b)d(a))$

$+a[d(b)d(b)d(a)]a+(d(b)^3]a[a^3]+(d(b)d(b)d(a))a^2$

$+[d(b)d(b)d(a)]ab+([d(b)d(a)d(b)]aa+a[d(b)d(a)b]d(a)$

$+a[d(a)d(b)d(a)]b+aa[d(a)d(b)d(b)]+a[d(a)d(b)b]d(a)$

$+a[bd(a)b][d(a)bd(a)]+ab[d(a)d(a)d(b)]+ab[(d(a)bd(a))bd(a)]+$

$ad(a)[bd(a)d(a)]+aa[d(a)d(a)d(b)]+ad(a)[bd(a)d(a)]=$

$[d(a)ba+ad(b)a+abd(a)][d(b)ab+bd(a)b+bad(b)][d(a)ba+ad(b)a+abd(a)]$

$=d(aba)d(bab)d(aba)=d(a)d(b)d(a)$.

Corollary 3.41 *If* $a \in \mathcal{Z}(S)$ *is an idempotent then :*

$$27aa(d(a)^3) = d(a)^3.$$

Proposition 3.42 *Let* a *be a strongly invertible element and* b *a strongly inverse of* a *and* d *a derivation on an additively regular semiring* S *of characteristic* $\neq 2$. *Then*

$$d(a)b1 + ad(b)1 = 0.$$

Definition 3.43 . *A subset* I *of a ternary semiring* S *is called a co-ideal of* S *if:*

1. *for any elements* a, b, c *in* I*;* $a.b.c \in I$*,*

2. *for any* $a \in I$ *and* $s \in S; a + s \in I$*.*

Clearly if I is a co-ideal of a ternary semiring S with 0, then $0 \in I$ if and only if $I = S$ and then any ideal which also is a co-ideal is S.

Example 3.44 *Let* $n \in \mathbb{Z}^-$*. The set* $I_n = \{a \in Z^- \ / \ a \leq n\}$ *is a co-ideal of* $\mathbb{Z}^-$*.*

Definition 3.45 *A co-ideal* I *of a ternary semiring* S *is called left cancelable if:*

$$a, b, abc \in I \implies c \in I$$

Definition 3.46 *A co-ideal* I *of a ternary semiring* S *is called right cancelable if:*

$$b, c, abc \in I \implies a \in I$$

Example 3.47 *In the ternary semiring* $S = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \ / \ a, b, c \in \mathbb{Z}^-\}$*; for any* $n \leq -1$*; the subset of* S*;* $\mathcal{I}_n = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \ / \ b \leq n \in\}$*; is a co-ideal.*

Definition 3.48 *A proper co-ideal* I *of a ternary semiring* S *is called prime if*

$$a + b \in I \implies a \in I \text{ or } b \in I$$

.

Example 3.49

1. *The co-ideal* $\mathcal{I}_n$ *of example 3.48 is prime if and only if* $n = -1$ *or* $n = 0$*.*
2. $d(\mathcal{I}_n) \subset \mathcal{I}_n$ *for the derivation of example 3.22.*

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