Dual Reciprocity-Boundary Element Method Applied to the Solution of the Equation of Helmholtz

Jorge Humberto Vargas Aparicio, Helvio Ricardo Mollinedo Ponce de León, Lesli Ortega Arroyo and José Ángel Ortega Herrera

***Abstract*— This paper develops the application of the Dual Reciprocity - Boundary Element Method (DR-BEM) to pose the problem of the Helmholtz equation in acoustics problems. Because one of the main problems in the analysis of acoustic vibrations in internal cavities is to determine the natural frequencies of the acoustic system and its associated vibration modes, in this paper we develop the formulation by the boundary element method (BEM) and two-dimensional geometries modal analysis to compare the results with the finite element method (FEM) modal analysis based on the DR-BEM which is self-developed method provides a good approximation compared with the results obtained by the FEM. The proposed procedure has been programmed using the MATLAB ® program. We have obtained the natural frequencies for a two-dimensional domain, comparing the results obtained using DR-BEM with the Finite Element Method.**

***Keywords*— Helmholtz Equation, Boundary Element Method (BEM), Finite Element Method (FEM) and Dual Reciprocity - Boundary Element Method (DR-BEM)**

I. INTRODUCTION

Various techniques to establish numerical auto-values problem of the Helmholtz equation. Among the most widespread numerical techniques to solve such problems are Finite Element Method (FEM) and the Boundary Element Method (BEM), the first is widely known for his versatility, but the BEM has some features that make it a very useful tool in certain problems.

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The FEM requires the entire domain partition therefore the number of algebraic equations to be solved increases with the size of the mesh, the BEM reduces the dimension of the problem using only the partition of the border, which leads to a considerable reduction in the number of equations and the resources used to generate the mesh [1] . While you cannot establish the superiority of one method over the other, currently the MEF takes further development and dissemination [2]. However, the current trend leads to utilize the advantages of both methods on a hybrid formulation [3].

The origin of the BEM is deeply linked to integral equations, however, we can highlight the work of Erich Trefftz [4], which based on the method proposed by Ritz (1908) states use test functions, that unlike the Ritz method, satisfies the differential equation in the whole domain but not necessarily the boundary conditions. Like other numerical methods, BEM began to develop with the advent of digital computers since 1960. The term Boundary Element Method first appeared in an article by Brebbia and Dominguez [5] published in 1977.

The treatment of the inertial terms and sources, as well as the treatment of auto-value problems using the Dual Reciprocity Method (DRM) developed by Nardini and Brebbia [6], allowed BEM applications expand to different areas of engineering.

One of the main problems in the analysis of acoustic vibrations in internal cavities is to determine the natural frequencies of the acoustic system and its associated vibration modes. In the present work develops through the BEM formulation and modal analysis in two-dimensional geometries to compare the results with the FEM modal analysis based on the DR-BEM which is self-developed method provides a good approximation compared with the results obtained by FEM [9]. Predicting noise and its control is becoming increasingly important in the design of a variety of systems.

II. MATHEMATICAL MODELING

A. *BEM formulation Helmholtz equation*

The equation governing the propagation of sound waves in an enclosure is given by the equation:

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0 \quad (1)$$

If the solution we seek is harmonic of the form $p = ue^{i\omega t}$ the equation can take the form known as Helmholtz equation [7]:

$$\nabla^2 u + k^2 u = 0 \quad (2)$$

Where ∇^2 is the Laplace operator, u = sound pressure, c = velocity of the fluid in the fluid medium, ω = angular frequency, $k^2 = \omega^2/c^2$ wave number.

Equation (1) in general is subject to the following boundary conditions:

$u = p_e$	sound pressure - Dirichlet condition	
$\frac{\partial u}{\partial n} = -i\omega\rho q_n$	normally speed - Neumann condition	(3)
$\frac{\partial u}{\partial n} = -i\omega\rho \frac{1}{Z_a} u$	Acoustical Impedance - Mixed condition	

Consider now, a domain Ω and boundary defined by $\Gamma = \Gamma_u \cup \Gamma_q$, where: Γ_u is defined the Dirichlet boundary condition and Γ_q Neumann condition as shown in Figure 1. The boundary integral equation for the equation (2) can be written using a residual weights formulation in the form:

$$\int_\Omega (\nabla^2 u + k^2 u)u^* d\Omega = \int_{\Gamma_q} (v - v_b)u^* d\Gamma - \int_{\Gamma_u} (u - u_b)q^* d\Gamma \quad (4)$$

Where u* is the fundamental solution of the Helmholtz equation q* its normal derivative.

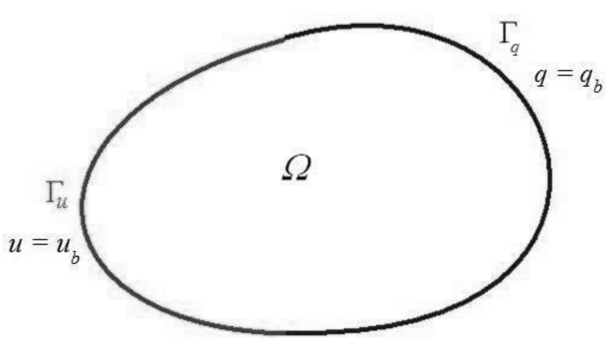


Fig.1. Domain Ω and boundaries Γ, u_b, q_b known values

Applying integration by parts and Gauss' theorem twice in (4) yields the equation:

$$\int_\Omega u(\nabla^2 u^* + k^2 u^*) d\Omega = -\int_{\Gamma_q} q_b u^* d\Gamma - \int_{\Gamma_u} q u^* d\Gamma + \int_{\Gamma_q} u q^* d\Gamma + \int_{\Gamma_u} u_b q^* d\Gamma \quad (5)$$

By definition, the fundamental solution of the Helmholtz equation can be found using a point source at the point *i* either domain, therefore u^* must satisfy:

$$\nabla^2 u^* + k^2 u^* + \delta_i = 0 \quad (6)$$

Where δ_i is the Dirac delta. Applying the properties of the Dirac delta can write (5) as:

$$u_i + \int_{\Gamma_u} u_b q^* d\Gamma + \int_{\Gamma_q} u q^* d\Gamma = \int_{\Gamma_u} q u^* d\Gamma + \int_{\Gamma_q} q_b u^* d\Gamma \quad (7)$$

Taking the point *i* in the boundary equation is obtained representation:

$$C_i u_i + \int_\Gamma u q^* d\Gamma = \int_\Gamma q u^* d\Gamma \quad (8)$$

Where the fundamental solution u^* and its normal derivative q^* for the Helmholtz equation in two dimensions are given by:

$$u^* = \frac{j}{4} H_0^1(kr) \quad (9)$$

$$q^* = -\frac{jk}{4} H_1^1(kr)\frac{\partial r}{\partial n} \quad (10)$$

H_0^1, H_1^1 are the Hankel functions of first kind of order zero and first-class, respectively.

Realizing the domain partition in boundary elements on equation (8) can be written in discrete form by:

$$C_i u_i + \sum_{j=1}^{N} \hat{H}_{ij} u_j = \sum_{j=1}^{N} G_{ij} q_j \quad (11)$$

Where $\hat{H} = \int_\Gamma q^* d\Gamma$, $G = \int_\Gamma u^* d\Gamma$

In matrix form:

$$[H]\{u\} - [G]\{q\} = 0 \quad (12)$$

B. Dual Reciprocity – Boundary Element Method

An alternative to using fundamental solutions of the Helmholtz equation is used Dual Reciprocity Method (DRM) [7], equation (2) can be written as:

$$\nabla^2 u = -k^2 u \tag{13}$$

Applying the same procedure as above, the BEM formulation of equation (12) yields:

$$[H]\{u\} - [G]\{q\} = k^2 \int_{\Omega} (u) u^* d\Omega \tag{14}$$

If right integral equation (14) it is possible to find a solution Ψ such that $\nabla^2 \Psi = u$ inside domain Ω, we can apply Gauss's theorem to obtain:

$$[H]\{u\} - [G]\{q\} = k^2 \left(-[G]\left\{\frac{\partial \Psi}{\partial n}\right\} + [H]\{\Psi\}\right) \tag{15}$$

Can now be set to an interpolation function approximating the pressure *u* within the domain:

$$u(x) = \sum_{m=1}^{M} C(x, \xi_m) \Phi(\xi_m) \tag{16}$$

Where x is a point in the domain, ξ_m is a point source at the boundary, Φ is a fictitious density function applied in ξ_m. The shape functions of most widely used have the form:

$$C(x, \xi_m) = R - r(x, \xi_m) \tag{17}$$

Where R is a constant (usually the longest distance between two points within the domain). Using these interpolation functions in (16) and solving the equation:

$$\nabla^2 \Psi = \sum_{m=1}^{M} \{R - r(x, \xi_m)\} \Phi(\xi_m) \tag{18}$$

Equation (18) can be solved for Ψ:

$$\Psi = -\sum_{m=1}^{M} \mathrm{D}(x, \xi_m) \Phi(\xi_m) \tag{19}$$

Where for two-dimensional problems:

$$\mathrm{D}(x, \xi_m) = \frac{r^3}{9} - \frac{Rr^2}{4}$$

Also:

$$\frac{\partial \Psi}{\partial n} = -\sum_{m=1}^{M} T(x, \xi_m) \Phi(\xi_m) \tag{20}$$

Where for two-dimensional problems:

$$T(x, \xi_m) = \frac{(3r)^2}{9} - \frac{2Rr}{4\frac{\partial r}{\partial n}}$$

Substituting the equation (15) can be written as:

$$[H]\{u\} - [G]\{q\} = k^2([G][T] + [H][D])\{\Phi\} \tag{21}$$

Equation (21) can be used (16) to form only in terms of sound pressure *u:*

$$[H]\{u\} - [G]\{q\} = k^2(-[G][T] + [H][D])[C]^{-1}\{\mathrm{u}\} \tag{22}$$

Equation (22) can then be written as:

$$[H]\{u\} - [G]\{q\} = k^2 [M]\{\mathrm{u}\} \tag{23}$$

Where $[M] = (-[G][T] + [H][D])[C]^{-1}$

Applying appropriate boundary conditions equation (23) may be a form of a generalized value problem:

$$[K]\{u\} = k^2 [M]\{u\} \tag{24}$$

Matrix obtained using the BEM is a matrix densely distributed and non-symmetrical, although it is possible to carry a symmetrical arrays through techniques such as modal synthesis is also possible to apply special algorithms for non-symmetric matrices as Lanczos [8] to solve the problems of non-symmetric matrices eigenvalues as those posed by the equation (24).

III. NUMERICAL SIMULATION

Software has been developed based on MATLAB ® to solve the Helmholtz equation in two-dimensional problems, applying the DR-BEM method. The program allows modal analysis (obtaining natural frequencies), using constant boundary elements to approximate the model geometry. The results were compared with the finite element program ANSYS®.

As an application example, a domain is considered a quadrilateral formed by rigid walls, as shown in Figure 2, the properties of the fluid medium (air) are:

Density = 1.23 kg/m^3

Speed of sound = 340 m/s

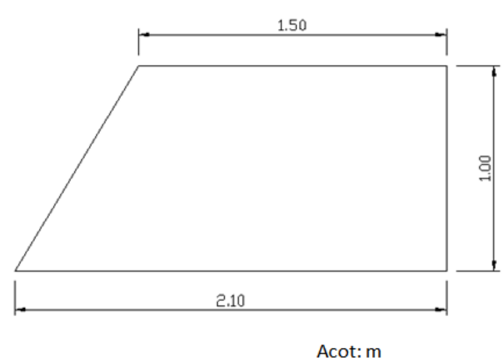


Fig. 2. Two-dimensional domain

Assumes the Neumann boundary condition uniform for all walls of the system.

Partitioning of the boundary using 16 constant boundary elements with a central node in the center of the element as shown in Figure 3.

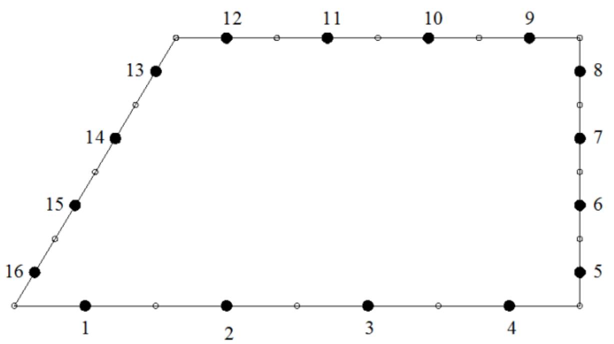


Fig.3. Partition the domain with constant boundary elements

For the analysis using the finite element method using 16 four-node quadrangular elements using the Ansys program. The finite element model consists of a total of 25 nodes, as shown in Figure 4.

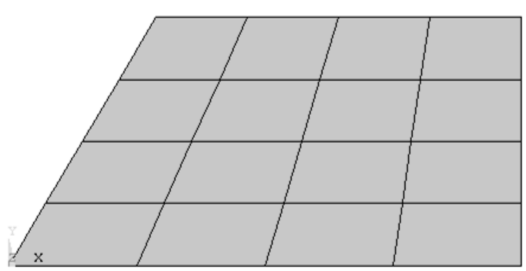


Fig.4. Partitioning Finite Element domain (Ansys Fluid29)

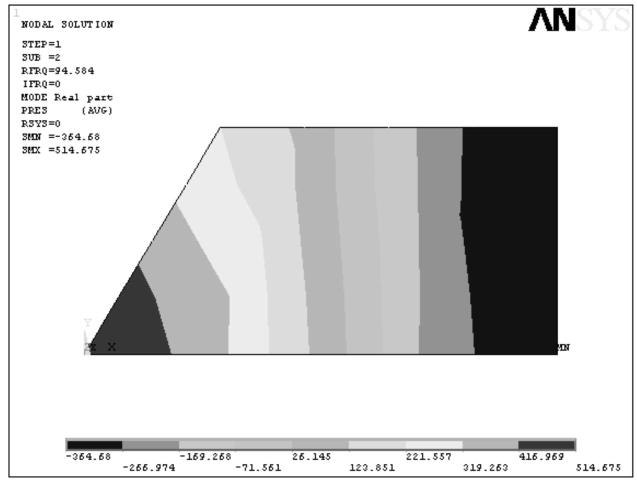
IV. RESULTS

Modal analysis was performed for the first 3 vibration modes. Table 1 summarizes the comparative results with each method.

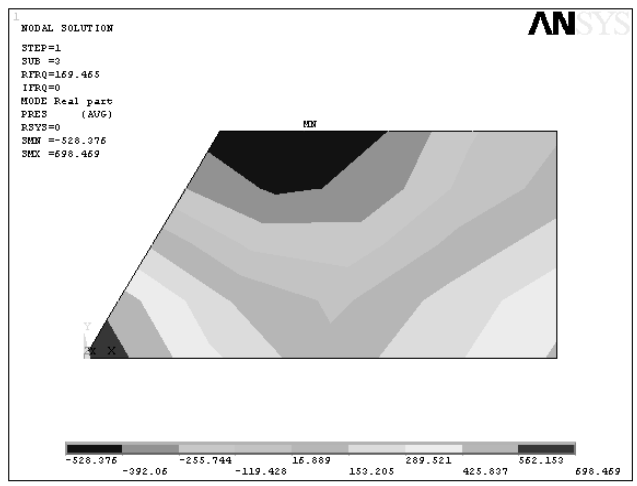
TABLE I
SUMMARIZES THE COMPARATIVE RESULTS WITH EACH METHOD

Mode	Frequency (Hz)	
	FEM	BEM
1	94.584	92.7
2	169.47	167.3
3	185.82	188.4

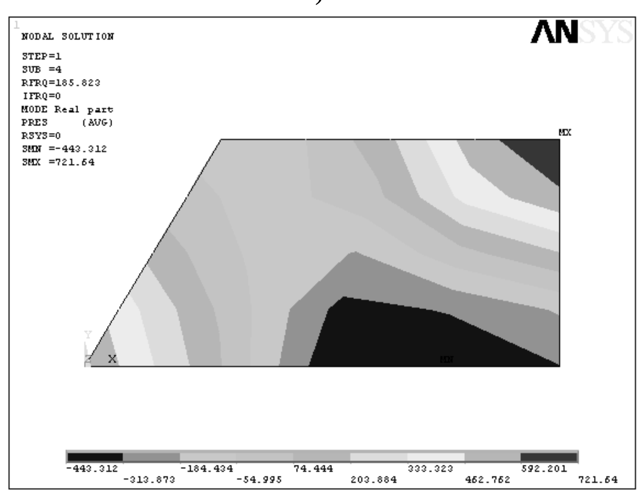
Graphs for the vibration modes are used Ansys ® post processor, the graphs for the first three modes of vibration are shown in Figures 5 a), b) and c).



5a)



5b)



5c)

Fig. 5. Vibration modes using FEM acoustic pressure a) 1st Mode, b) 2nd. Mode, c) 3rd Mode

V. CONCLUSIONS

The results obtained using the DR-BEM method have a good relationship with those obtained by the FEM. Is taken as an application example fairly simple two-dimensional geometry with the purpose of evaluating the formulation of DR - BEM and programmed in MATLAB ® software. The formulation presented allows to extend its application to three-dimensional problems.

Used boundary constant type elements (one node per element), it can be seen that they provide very good approximation for such problems. BEM an obvious advantage over the finite element method is the smallest number of unknowns, thus a smaller size of the matrices which can represent significant savings in computational larger problems.

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