Option Pricing of an Asset with Seasonal and Periodic Supply

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Abstract— **In option pricing the rate of change of asset price with time can be viewed to be directly proportional to the Walrasian [6] excess demand. Scholars such as Jacques [3] and Onyango [4], have used the excess demand concept with linearised demand and supply functions to derive and solve both deterministic and stochastic logistic differential equations for stock price. The underlying assets in option pricing are unique and can be seasonal and periodic like for electricity, water and other 'weather' derivatives. In this paper we develop and solve both deterministic and stochastic logistic differential equations for option pricing using the excess demand concept but with a linear demand function and a seasonal and periodic supply function.**

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I. INTRODUCTION

An option is a contract that gives the holder a right to buy or sell an underlying asset. The underlying assets include stocks, stock indices, foreign currencies, futures contracts, interest rates amongst other financial derivatives. A call option gives the holder the right to buy, while the put option gives the holder the right to sell the underlying asset by a certain date for a given price. The price at maturity is called the strike price. American options are those that can be exercised at any time up to the maturity date, while European options can only be exercised on the maturity date [4].

The law of demand states that the quantity of an asset that buyers are able and willing to buy in a given time increases as the price of an asset decreases, while the law of supply states that as the price of an asset increases, the quantity of an asset that sellers are able and willing to sell increases in a given time, all factors constant [3]. The interest of traders and investors is always to know the asset price at any time and the equilibrium price.

II. DETERMINISTIC MODELS

Walras [6] proposed that market equilibrium could be realized through a price adjustment process in which consumers and producers interact through a central `auctioneer' who adjusts the general price levels towards the equilibrium price based on excess demand. Since trading takes place continuously and prices are also adjusted continuously per unit time, Samuelson [5] derived a family of differential equations based on this excess demand of the form;

$$\frac{dS_i(t)}{dt} = K_i E[S_i(t)] = K_i\{Q_d[S_i(t)] - Q_s[S_i(t)]\} \qquad (1)$$

where $i = 1, 2 \ldots\ldots n$, K_i is a positive rate of adjustment of the market to changes in supply and demand, $S_i(t)$ is the asset price at time t, Q_d is the quantity demanded, Q_s is the quantity supplied and $E[S_i(t)]$ is the excess demand function given by $\{Q_d[S_i(t)] - Q_s[S_i(t)]\}$.

When supply exceeds demand (surplus or excess supply), $\frac{dS(t)}{dt}$ is negative and the price reduces. When the demand exceeds supply (shortage or excess demand), $\frac{dS(t)}{dt}$ is positive and prices increase. When supply equals demand, $\frac{dS(t)}{dt} = 0$ and the price is at market equilibrium leading to an equilibrium price (S^*) such that;

$$Q_d[S^*(t)] = Q_s[S^*(t)]$$

From equation (1) we have a particular fractional rate of increase of asset price proportional to excess demand given as;

$$\frac{1}{S(t)}\frac{dS(t)}{dt} = K\{E[S(t)]\} = K\{Q_d[S(t)] - Q_s[S(t)]\}, \qquad (2)$$

where K is a constant and $E[S(t)]$ is the excess demand function.

Considering linear demand and supply functions about the equilibrium, let

$Q_d[S(t)] = \alpha[S^* - S(t)]$ and

$Q_s[S(t)] = -\beta[S^* - S(t)]$ where α and β are constants, $Q_d[S(t)]$ is a decreasing linear function of $S(t)$, $\alpha > 0$, $Q_s[S(t)]$ is an increasing linear function of $S(t)$ and $\beta > 0$. Substituting these linear demand and supply functions about the equilibrium in equation (2) we have a deterministic Velhurst logistic first order ordinary differential equation;

$$\frac{dS(t)}{dt} = rS(t)[S^* - S(t)] \qquad (3)$$

where $r = K(\alpha + \beta)$ is the asset price growth rate and if $S^* = S(t)$ then $r = 0$.
Solving equation (3) given the initial condition that at time $t = 0$, $S(0) = S_0$ the asset price;

$$S(t) = \frac{S_0 S^*}{(S^* - S_0)e^{rS^*(t-t_0)} + S_0}$$

where $r, t \geq t_0$ and as $t \to \infty$, $S(t) \to S^*$ [4].

If we let demand be positive and a linear monotonically decreasing function of price of the form;

$$Q_d S(t) = a - bS(t), 0 < S(t) < \frac{a}{b} \qquad (4)$$

and supply (purely a function of time (t))
to be a seasonal and periodic function of the form;

$$Q_S = c(1 - \cos \alpha t), t \geq 0 \qquad (5)$$

where a, b, c and α are constants.
When we substitute both equation (4) and (5) in the excess demand function equation (1), and with μ replacing K we derive a periodic deterministic logistic first order differential equation of the form;

$$\frac{dS(t)}{dt} = \mu[a - bS(t) - c(1 - \cos\ \alpha t)] \qquad (6)$$

Rearranging equation (6) we get;

$$\frac{dS(t)}{dt} + \mu b S(t) = \mu[a - c + \cos\ \alpha t)] \qquad (7)$$

Using the integrating factor method, let;

$$I(t) = e^{\int \mu b dt} = e^{\mu b t + c_2} = e^{\mu b t} + e^{c_2}$$

where c_2 is a constant of integration.

For a homogeneous solution, $e^{c_2} = 0$, which implies that $I(t) = e^{\mu b t}$, therefore;

$$S(t) = e^{-\mu b t} \int e^{\mu b t}[\mu(a - c + cos\alpha t)dt + c_1] \qquad (8)$$

Expanding and integrating equation (8) term by term we have;

$$S(t) = \frac{\mu(a-c)}{\mu b} + \mu c e^{-\mu b t} \int e^{\mu b t} cos\alpha t dt + c_1 e^{-\mu b t} \qquad (9)$$

Evaluating the integral on the right hand side of equation (9) gives us;

$$\int e^{\mu b t} cos\alpha t dt = \frac{e^{\mu b t}}{\mu^2 b^2 + \alpha^2}(\mu b cos\alpha t + \alpha \sin \alpha t) + c_3 \qquad (10)$$

where c_3 is a constant of integration. Substituting equation (10) in (9), simplifying and merging the constants of integration to c_1 we have;

$$S(t) = \frac{a - c}{b} + \frac{\mu c}{\mu^2 b^2 + \alpha^2}(\mu b cos\alpha t + \alpha sin\alpha t) + c_1 e^{-\mu b t} \qquad (11)$$

Taking the initial conditions; at $t = 0$, $S(t) = S_0$ and making c_1 the subject of formulae;

$$c_1 = S_0 - \frac{a - c}{b} - \frac{\mu^2 bc}{\mu^2 b^2 + \alpha^2}$$

Substituting c_1 in equation (11) we get the homogeneous solution;

$$S(t) = \left(S_0 - \frac{a - c}{b} - \frac{\mu^2 bc}{\mu^2 b^2 + \alpha^2}\right)e^{-\mu b t} + \frac{a - c}{b} + \frac{\mu c}{\mu^2 b^2 + \alpha^2}(\mu b cos\ \alpha t + \alpha sin\alpha t), \mu, t \geq 0 \qquad (12)$$

The last term of equation (12) can be simplified to;

$$\frac{\mu c}{\mu^2 b^2 + \alpha^2}(\mu b cos\alpha t + \alpha sin\alpha t) = \frac{\mu c}{\sqrt{\mu^2 b^2 + \alpha^2}} \sin(\alpha t + \theta)$$

where $\theta = \tan^{-1}\frac{\mu b}{\alpha}$

Hence the homogenous solution (equation 12) can also be written as;

$$S(t) = \left(S_0 - \frac{a - c}{b} - \frac{\mu^2 bc}{\mu^2 b^2 + \alpha^2}\right)e^{-\mu b t} + \frac{a - c}{b} + \frac{\mu c}{\sqrt{\mu^2 b^2 + \alpha^2}} \sin(\alpha t + \theta), \qquad \mu, t \geq 0 \qquad (13)$$

Note that:

$$\lim_{t\to\infty}\left(S_0 - \frac{a - c}{b} - \frac{\mu^2 bc}{\mu^2 b^2 + \alpha^2}\right)e^{-\mu b t} \to 0$$

and equation (13) becomes;

$$S(t) \approx \frac{a - c}{b} + \frac{\mu c}{\sqrt{\mu^2 b^2 + \alpha^2}} \sin(\alpha t + \theta)$$

where the equilibrium price, $S^* = \frac{a-c}{b}$

If we let $P = \frac{\mu c}{\sqrt{\mu^2 b^2 + \alpha^2}} \sin(\alpha t + \theta)$

The key features of this periodic deterministic logistic equation with a positive growth rate is that, first the values of $S(t)$ oscillate with an amplitude of $\frac{\mu c}{\sqrt{\mu^2 b^2 + \alpha^2}}$ and a period of $\frac{360}{\alpha t}$, secondly for all $S(0 > 0)$, $\lim_{t\to\infty} S(t) = S^* \mp P$ and finally if $S(0) = S^*$, then $S(t) = S^*$ for all $t > 0$.

III. STOCHASTIC MODELS

Black, Scholes and Merton [1] were the pioneer scholars in the use of stochastic models in option pricing. They used the geometric Brownian motion given by the equation;

$$dS(t) = \mu S(t)dt + \sigma S(t)dZ(t)$$

where $S(t)$ is the asset price at time t, μ is the drift, σ is the volatility and $dZ(t)$ is the standard Wiener process (that is $dZ(t) = \epsilon\sqrt{dt}$ and $\epsilon \sim N(0,1)$) to find a theoretical estimate of the price of European-style options. This model has been modified and applied by many other scholars and practitioners since 1973. [2], [7].

Considering the stochastic effects in the Walrasian market equilibrium Onyango [4] modified equation (3) as follows;

$$\frac{1}{S(t)}\frac{dS(t)}{dt} = h(\alpha + \delta\alpha + \beta + \delta\beta)(S^* - S(t)) \qquad (14)$$

where α and β are the elasticity of supply and demand respectively, $\delta\alpha$ and $\delta\beta$ are cumulative random shocks during a trading period dt and h is a constant.

Applying Wiener process conditions, let $h(\delta\alpha + \delta\beta) = \sigma dZ$ and $\mu = h(\delta\alpha + \delta\beta)$ where σ is the volatility of the underlying asset price and μ is the drift rate. Equation (14) becomes a stochastic logistic Brownian motion equation for the asset price given by;

$$\frac{1}{S(t)}\frac{dS(t)}{(S^* - S(t))} = \mu dt + \sigma dZ(t) \qquad (15)$$

where $S^* \neq S(t)$

The solution to equation (15) is;

$$S(t) = \frac{S_0 S^*}{S_0 + (S^* - S_0)exp^{-(\mu S^*(t-t_0) + \sigma S^* Z(t))}}$$

where $r, t \geq 0$. [4].

The stochastic counterpart of equation (6) is;

$$\frac{dS(t)}{(a - bS(t) - c(1 - \cos\alpha t))} = \mu dt + \sigma dZ(t) \qquad (16)$$

Let $Y = (a - bS(t) - c(1 - \cos\alpha t))$, then substituting for Y in equation (16) we have;

$$dS(t) = \mu Y dt + \sigma Y dZ(t) \qquad (17)$$

Note that $\frac{dY}{dS(t)} = -b$, therefore $dS(t) = \frac{dy}{-b}$ hence equation (17) can be written as;

$$dY = -b\mu Y dt - b\sigma Y dZ(t) \qquad (18)$$

Integrating equation (18) in the time interval $t_0 \leq t_i$ we have;

$$\ln Y(t_i) - \ln Y(t_0) = -b\mu(t_i - t_0) - b\sigma Z(t_i), \qquad Z(0) = 0$$

which can be simplified to;

$$Y(t) = Y_0 exp^{-b[\mu(t_i - t_0) + \sigma Z(t_i)]}, \; Y(t_0) = Y_0 \qquad (19)$$

Taking $t = 0$, $Y_0 = a - b(S_0)$ and substituting for Y and Y_0 in equation (19) we have;

$$a - bS(t) - c + c\cos\alpha t = (a - b(S_0))exp^{-b[\mu(t_i - t_0) + \sigma Z(t_i)]} \qquad (20)$$

Making $S(t)$ the subject of formulae equation (20) becomes;

$$S(t) = \frac{a - c}{b} + \frac{c\cos\alpha t}{b} - \frac{(a - b(S_0))exp^{-b[\mu(t_i - t_0) + \sigma Z(t_i)]}}{b} \qquad (21)$$

where $\mu, t \geq 0$.

Note that $S(t) \approx S^* \mp \frac{c\cos\alpha t}{b}$ as $t \to \infty$ where the equilibrium price $S^* = \frac{a-b}{c}$

IV. CONCLUSION

We have succeeded in formulating and solving both deterministic and stochastic logistic differential equations for option pricing of assets whose supply is seasonal and periodic using the excess demand function. Although we have assumed that asset supply is purely a function of time and takes a periodic form with a constant amplitude and period this may not always be true.

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