Norms of Tensor Products Elementary Operators

[1]Beatrice A. Odero, [2]John A. Ogonji and [3]G. K. Rao
[1,2,3]School of Mathematics, Statistics and Actuarial Science, P. O. Box Private Bag, Maseno, Kenya
[1]adhiambobetty@rocketmail.com, [2]johnagure@maseno.ac.ke

Abstract—**In this paper, we determine the norm of a two-sided symmetric operator in an algebra. More precisely, we investigate the lower bound of the operator using the injective tensor norm. Further, we determine the norm of the inner derivation on irreducible C*- algebra and confirm Stamfli's result for these algebras.**

Mathematics Subject Classification: Primary 46L57; Secondary 47A30, 47A80

***Keywords*— Bounded Operators, Derivation, Two Sided symmetrised Operator and Algebra**

I. THE NORM OF A DERIVATION

We determine the norm of the inner derivation Δ_T : TA – AT acting on B(H) which is irreducible. More precisely, we show that $\|T_{A,A}\| = 2\, inf\{\|A - \lambda\| \lambda \epsilon \mathbb{C}\}$. A derivation Δ on a C*-algebra is a linear mapping $\Delta: \mathcal{A} \to \mathcal{A}$ satisfying the usual Leibniz product rule i.e.$\Delta(x, y) = x(\Delta y) + (\Delta x)y \; \forall\, x, y \in \mathcal{A}$. Such a mapping is bounded as was first shown by Sakai [16]. If there is an element a such that $\Delta x = xa - ax \; \forall\, x \in \mathcal{A}$, then the derivation is **inner**. In most cases such an element doesn't exists in $\mathcal{A}$. Therefore one tries to extend the derivation Δ to a bigger C*-algebra which may contain an implementing element.
Since Δ is inner, it is easier to estimate its norm which of course, is important from the analytic point of view. It is easy to see that if $\Delta x = xa - ax \, \forall x \in \mathcal{A}$, then $\|\Delta\| \leq 2dist(a, Z(\mathcal{A}))$ where $\mathcal{Z}(\mathcal{A})$ is the cener of$\mathcal{A}$.

II. PRELIMINARY RESULTS

"We say that a state f of a C* -algebra B(H) is definite on the self-adjoint operator A in B(H) when $f(A^2) = f(A)^2$. In this case, f is multiplicative on the C*- subalgebra generated by A. The following lemma is a combination of Singer's argument that the derivations of commutative C* -algebras are 0 and results on the multiplicative properties of definite states". See [7].

2.1 Lemma

If Δ is a derivation of the C- algebra B(H) and f is definite on A in B(H), then* $f(\Delta(A)) = 0$

Proof.

We note that $\Delta(I) = \Delta(I^2) = 2\Delta(I)$, so that $\Delta(I) = 0$. Thus $\Delta(A) = \Delta(A - f(A)I$; and we assume that $f(A) = 0$. In this case $0 = f(A^+) = f(A^-)$, where $A = A^+ - A^-$, A^+ and A^- "positive" and "negative" parts of A; for $A^+A = {A^+}^2$, so that $0 = f(A^+)f(A) = f(A^+A) = f({A^+}^2) = f(A^+)^2$. Since $\Delta(A) = \Delta(A^+) - \Delta(A^-)$, it will suffice to show that $A > 0$ and $f(A) = 0$. Let $T = A^{\frac{1}{2}}$. Then$f(T) = 0$. Hence $f(\Delta(A)) = f|\Delta(T)T| + f|T\Delta(T)| = f|\Delta(T)|f(T) + f(T)f|\Delta(T)| = 0$. The substance of the foregoing lemma is that each derivation of a C *- algebra maps each self-adjoint operator in the algebra onto an operator that has 0 diagonal relative to a diagonalization which diagonalizes A [7].

2.2 Theorem

Each derivation of a C-algebra annihilates its center [7].*

Proof.

Let Δ be a derivation of the C* -algebra B(H) with center$Z(B(H))$. Let f be a pure state of B(H), and z an element of $Z(B(H))$. The representation of B(H) associated with f is irreducible [23] and therefore maps $Z(B(H))$ into scalars. Together with the Schwarz inequality, this yields that $f$is multiplicative on $Z(B(H))$. From the preceding lemma, $f(\Delta(z)) = 0$. Since the pure states of B(H) separate B(H), $\Delta(z) = 0$.

2.3 Lemma

If Δ is a derivation of the C-algebra B(H) acting on the space H, then Δ has a unique ultra weakly continuous extension which is a derivation of $B(H)^-$.*

Proof.

The positive operators in the unit ballυ_1 of B(H). Now $A \to ([A\Delta(A) + \Delta(A)A]x, y) \quad (= (\Delta A^2)x, y))$ is strongly continuous at 0 on υ_{1^*}, the set of self-adjoint operators in the unit ball of B(H), since $|\langle (A\Delta(A) + \Delta(A)A)x, y\rangle| \leq \|\Delta\|(\|Ax\|\|y\| + \|x\|\|Ay\|)$ where $\|\Delta < \infty\|$ by Sakai's theorem [21]. Moreover, $A \to A^{\frac{1}{2}}$ is strongly continuous at 0 on positive operators, since $\left\|A^{\frac{1}{2}}\right\| = |\langle Ax, x\rangle| \leq \|Ax\|\|x\|$. Thus $A \to A^{\frac{1}{2}} \to (\Delta(A)x, y)$ is strongly continuous at 0 on

ϑ_1. We note next that Δ is weakly continuous on ϑ_1 to B(H) in the weak operator topology. Since $Ax = A^+x - A^-x$ with A^+ and A^- orthogonal, $\|A^+\| \le \|Ax\|$ and $\|A^-\| \le \|Ax\|$; so that $A \to A^+$ and $A \to A^-$ are strongly continuous mappings on the self-adjoint operators in B(H) at 0. Thus $A \to (\Delta(A^+)x, y) - (\Delta(A^-)x, y = (\Delta(A)x, y)$ is strongly continuous at 0 on ϑ_{1^*}. By linearity this mapping is strongly continuous at 0 on $2\vartheta_{1^*}$ and from this , everywhere on ϑ_{1^*}. Hence the inverse image of a closed convex subset of the complex numbers under $A \to (\Delta(A)x, y)$ has an intersection with ϑ_{1^*} which is strongly closed relative to ϑ_{1^*}. This intersection being convex, each weak weak limit point is a strong limit point [3, 5], so that it is weakly closed relative to ϑ_{1^*}. Since the closed convex subsets of the complex numbers form a subbase for the closed subsets, $A \to (\Delta(A)x, y)$ is weakly continuous on ϑ_{1^*}. Now $A \to \frac{(A+A^*)}{2}$ and $A \to \frac{(A-A^*)}{2}$ are weakly continuous mappings of ϑ_1 into ϑ_{1^*}; so that $A \to (\Delta\left(\frac{A+A^*}{2}\right)x, y + i(\Delta\left(\frac{A-A^*}{2i}\right)x, y = (\Delta(A)x, y)$ is weakly continuous on ϑ_1 . The linearity of Δ yields its uniform continuity relative to the weak-operator uniform structure on ϑ_1. From Kaplansky density theorem [14], $\vartheta_{\bar{1}}$ is the unit ball in $B(H)^-$, and is compact in the weak-operator topology. Thus Δ has a unique weak- operator continuous extension to $\vartheta_{\bar{1}}$, and this extension has an obvious extension Δ from $\vartheta_{\bar{1}}$ to $B(H)^-$. It is easily checked that this extension is well defined and linear. For if $x \in H$, $(AT) \to ([\bar{\Delta}(AT) - \bar{\Delta}(A)T - A\bar{\Delta}(T)]x.x)$ is strongly continuous on $\vartheta_{\overline{1^*}} \times \vartheta_{\overline{1^*}}$, by strong continuity of operator multiplication on bounded sets, weak continuity of $\bar{\Delta}$ on $\vartheta_{\bar{1}}$ and boundedness of Δ (hence $\bar{\Delta}$). Since this mapping is 0 on $\vartheta_{1^*} \times \vartheta_{1^*}$, a strongly dense subset of $\vartheta_{\overline{1^*}} \times \vartheta$; it is 0 on $\vartheta_{\overline{1^*}} \times \vartheta_{\overline{1^*}}$, for each x, so that $\bar{\Delta}$ is a derivation on $B(H)^-$ [7].

2.4 Lemma

Every derivation Δ on a C^*-algebra is bounded.

Proof.

Since every derivation on a non-unital C^*-algebra can be uniquely extended to its minimal unitization, the assertion follows from the fact that every generalised derivation on a unital C^*-algebra is bounded.

III. MAIN RESULTS

3.1 Lemma

Every derivationΔ on a C^*-algebra $\mathcal{A}$ vanishes on the center $Z(\mathcal{A})$ of .

Proof.

Let $a \in Z(\mathcal{A})$. Then for all $x \in \mathcal{A}, x(\Delta a) = \Delta(xa) - (\Delta x)a = \Delta(ax) - a(\Delta x) = (\Delta a)x$ where $\Delta a \in Z(\mathcal{A})$. From $a(\Delta a) - (\Delta a)a = 0$, "the boundedness of a derivation and the general version of Kleinecke-Shirokov theorem" [7], we conclude that Δa is quasinilpotent but being central, this implies that $\Delta a = 0$.

3.2 Lemma

If $\|T\| = \|x\| = 1$ and $\|T\|^2 \ge (1-\varepsilon)$, then $\|(T^*T - Ix2 \le 2\varepsilon$.

Proof.

$$0 \le \|(T^*T - I)x\|^2$$
$$\|(T^*T - I)x\|^2 = \langle (T^*T - I)x, (T^*T - I)x\rangle$$
$$= \langle T^*Tx - Ix, T^*Tx - Ix\rangle$$
$$= \langle T^*Tx, T^*Tx\rangle - \langle T^*Tx, Ix\rangle - \langle Ix, T^*Tx\rangle + \langle Ix, Ix\rangle$$
$$= \|T^*Tx\|^2 - 2\langle Tx, Tx\rangle + \|x\|^2$$
$$= \|T^*Tx\|^2 - 2\|Tx\|^2 + \|x\|^2$$
$$\le (\|T^*\|\|T\|\|x\|)^2 - 2\|Tx\|^2 + \|x\|^2 = 1 - 2\|Tx\|^2 + 1$$
$$= 2(1 - \|Tx\|^2) \le 2(1-(1-\varepsilon) = 2\varepsilon.$$

3.3 Lemma

Let$\mu \in W(T)$. Then $\Delta_T \ge 2(\|T\|^2 - |\mu|^2)^{\frac{1}{2}}$.

Proof.

We note that $\|\Delta_T\| = \sup\{\|TA - AT\| : A \in B(H), \|A\| = 1\}$. Since $\mu \in W(T)$, there exists $x_n \in H$ such that $\|x_n\| = 1, \|Tx_n\| \to \|T\|$, and $(Tx_n, x_n) \to \mu$. If we set $Tx_n = \alpha_n x_n + \beta_n y_n$, where $\langle x_n, y_n\rangle = 0$ and $\|y_n\| = 1$. Also, $V_n x_n = x_n, V_n y_n = -y_n$ and $V_n = 0$ on $\{x_n, y_n\}$. Then $\|(TV_n - V_nT)x_n\|^2 = \|Tx_n - V_nTx_n\|^2 = \|\alpha_n x_n + \beta_n y_n - Vn\alpha nxn + \beta nyn2$

$$= \langle \alpha_n x_n + \beta_n y_n - V_n(\alpha_n x_n + \beta_n y_n), \alpha_n x_n + \beta_n y_n - V_n(\alpha_n x_n + \beta_n y_{n)}\rangle$$
$$= \langle \alpha_n x_n + \beta_n y_n, \alpha_n x_n + \beta_n y_n\rangle - \langle \alpha_n x_n + \beta_n y_n, V_n(\alpha_n x_n + \beta_n y_n)\rangle - \langle V_n(\alpha_n x_n + \beta_n y_n), \alpha_n x_n + \beta_n y_n\rangle + \langle V_n(\alpha_n x_n + \beta_n y_n), V_n(\alpha_n x_n + \beta_n y_n)\rangle$$

3.4. Theorem

$\|\Delta_T\| = 2\|T\|$ if and only if $0 \in W(T)$.

Proof.

From lemma 2.3, we have that $\|\Delta_T\| \ge 2\|T\|$ if $0 \in W(T)$. Since $\|\Delta_T\| \le 2\|T\|$ for any T, sufficiency is proved. We assume that the $\|\Delta_T\| = 2\|T\|$, and hence there exists x_n and A_n such that $\|x_n\| = \|A_n\| = 1$ and$\|(TA_n - A_nT)x_n\| \to 2\|T\|$. Clearly, $\|A_nx_n\| \to 1, \|Tx_n\| \to \|T\|$ and $\|TA_nx_n\| \to \|T\|$. Moreover, since $\|(TA_n - A_nT)x_n\| \to 2\|T\|$, $TA_nx_n = -A_nTx_n + \vec{\varepsilon}_n$ where$\|\vec{\varepsilon}_n\| \to 0$. Let $(Tx_n, x_n) \to \mu$ by choosing subsequence if necessary, i.e. $\mu \in W(T)$. We observe that $(TA_nx_n, A_nx_n) = -(A_nTx_{n,}A_n x_n + \varepsilon_n = -(Tx_n, A^*{}_nA_nx_n = -(Tx_{n,}x_n) + \varepsilon'_n$ where the last step follows from lemma 2.2. Thus, $\lim_{n\to\infty}(TA_n x_n, A_nx_n) = -\mu$. Since $\mu, -\mu \in W(T)$, it follows that $0 \in W(T)$.

3.5. Theorem

If $0 \in W(T)$, *then* $\|T\|^2 + |\lambda|^2 \leq \|T + \lambda\|^2 \forall \in \mathbb{C}$. *Conversely, if* $\|T\| \leq \|T + \lambda\| \; \forall \; \lambda \in \mathbb{C}$, *then* $0 \in W(T)$.

Proof.

If $0 \in W(T)$, then there exists $x_n \in H, \|x_n\| = 1$ such that $\|(T + \lambda)x_n\|^2 = \|Tx_n\|^2 + Re\bar{\lambda}(Tx_n, x_n) + |\lambda|^2 \to \|T\|^2 + |\lambda|^2$. Conversely, let $\|T\| \leq \|T + \lambda\| \; \forall \lambda \in \mathbb{C}$. We assume that $0 \notin W(T)$. By rotating T, we assume that $ReW(T) \geq \tau > 0$. Let $\zeta = \{x \in H: \|x\| = 1$ and $Re(Tx, x) \leq {}^{\tau}/_{2}\}$, $\eta = \sup\{\|Tx\|: x \in \zeta\}$. Then $\eta \leq \|T\|$. Let $\mu = \min\{{}^{\tau}/_{2}, (\|T\| - {}^{\tau}/_{2}\}$ and consider$(T - \mu)$. If $x \in \zeta$, Then $\|(T - \mu)x\| \leq \|Tx\| + \mu \leq \eta + \mu < \|T\|$. Let $Tx = (a + ib)x + y$ where $x \notin \zeta, \|x\| = 1$ and $(x, y) = 0$. Then $\|(T - \mu)x\|^2 = (a - \mu)^2 + b^2 + \|y\|^2 = \|Tx\|^2 + (\mu^2 - 2a\mu) < \|T\|^2$ since $a > \mu > 0$ i.e. $\|T - \mu\| < \|T\|$, contrary to the hypothesis.

3.6 Corollary (Pythagorean relation for operator)

Ⲁnd only if $\lambda = z_\circ$.

Proof.

Now, there exists $z_\circ \in \mathbb{C}$ such that $\|T - z_\circ\| \leq \|(T - z_\circ) + \lambda\| \; \forall \lambda \in \mathbb{C}$. The rest of the proof easily follow from theorem 2.5.

3.7. Theorem

Let Δ_T *be a derivation on* $B(H)$. *Then* $\|\Delta_{T/B(H)}\| = \sup\{\|TA - AT\|: A \in B(H), \|A\| = 1\} = inf_{\lambda \in \mathbb{C}}\{2\|T - \lambda\|\}$.

Proof.

Since$\|TA - AT\| = \|(T - \lambda)A - A(T - \lambda\| \leq 2\|T - \lambda\|\|A\|$. It follows therefore that $\|\Delta_T\| \leq inf_{\lambda \in \mathbb{C}}\{2\|T - \lambda$. On the other hand, $T{-}\lambda$ is larger for λ large. So $infT{-}\lambda$ must be taken on at some point, say $Z_\circ$. But $\|T - Z_\circ\| \leq \|(T - Z_\circ + \lambda\| \; \forall \lambda \in \mathbb{C}$ implies that $0 \in W(T - Z_\circ)$. Hence $\|\Delta_T\| = \|\Delta_{T - Z_\circ}\| = 2\|T - Z_\circ\|$.■

3.8. Definition

A C*-algebra $\mathcal{A}$ is irreducible if the commutant of $\mathcal{A}$ contains only the scalars.

3.9. Theorem

Let B(H) be an irreducible C-algebra on H. Let* $T \in B(H)$. *Then* $\|\Delta_{T/B(H)}\| = \sup\{\|TA - AT\|: A \in B(H), \|A\| = 1\} = inf_{\lambda \in \mathbb{C}}\{2\|T - \lambda\|$.
See [19] for the proof.

3.10. Theorem

Let A, B $\in$ *B(H). Then* $\|T_{A,B}\| = \sup\{\|AX - XB\|: X \in$ *BH, X=1=infλ∈ℂA−λ+B−λ.*

Proof.

$\|T_{\boldsymbol{A,B}}\| \leq \inf\{\|A - \lambda\| + \|B - \lambda\|\}$ follows from theorem 3.3.7. If we let $inf_{\lambda \in \mathbb{C}}\{\|A - \lambda\| + \|B - \lambda\| = \|A - \lambda_\circ\| + \|B - \lambda_\circ\|$. Then it follows from [19] lemma 6 and theorem 7 that $\|T_{A,B}\| = \|T_{A - \lambda_\circ, B - \lambda_\circ}\| = \|A - \lambda_\circ\| + \|B - \lambda_\circ\|$. If $A = B$, then the norm of $T_{A,B}$ is an inner derivation induced by A or B respectively i.e. $\|T_{A,A}\| = \inf\{\|A - \lambda\| + \|A - \lambda\|: \lambda \in$ *ℂ=2*inf*A−λ:λ∈ℂ=2RA* where *RA* is the radius of the spectrum of A. If $B(H)$ is irreducible then $\|T_{A,A}\| = 2\inf\{\|A - \lambda\|: \lambda \in \mathbb{C}\}$ implies that λ is the center of $B(H)$. Further if X is close to λ, then the norm is small hence X almost commute with the elements of the unit ball of $B(H)$.

IV. CONCLUSION

We have shown that the constant $c = 2$ i.e. $\|T_{a,b}\| \geq 2\|a\|\|b\|$ by taking $T_{a,b} = a \otimes b + b \otimes a$. We have also shown that $\|T_{A,B}\| = inf_{\lambda \in c}\{\|A - \lambda\| + \|B - \lambda\|\}$ which in turn is an inner derivation when A coincides with B.

ACKNOWLEDGMENT

Special thanks to Prof. Agure John of Maseno University, and Prof. G. K. Rao for their valuable comments and guidance towards this paper.

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Beatrice Adhiambo Odero, MSc (Pure Mathematics) Graduate Student, Maseno University, School of Mathematics, Statistics and Actuarial Science, Maseno, Kenya.

Prof. Agure John, PhD (Pure Mathematics), Professor, School of Mathematics, Statistics and Actuarial Science, Maseno, Kenya.

Prof. G. K. Rao, PhD (Pure Mathematics), Professor, School of Mathematics, Statistics and Actuarial Science, Maseno, Kenya.