Suborbital Graphs and their Properties for Unordered Triples in A_n (n=5,6,7) Through Rank and Subdegree Determination

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Abstract– **In this paper, through computation of the rank and subdegrees of alternating group A_n (n=5,6,7) on unordered triples we construct the suborbital graphs corresponding to the suborbits of these triples. When A_n ($n \geq 5$) acts on unordered pairs the suborbital graphs corresponding to the non-trivial suborbits are found to be connected, regular and have undirected edge except when n=6. Further, we investigate properties of the suborbital graphs constructed.**

Mathematics Subject Classification: **Primary 05E18; Secondary 05E30, 14N10, 05E15**

Keywords– **Rank, Subdegrees, Unordered Triple of an Alternating Group and Suborbital Graphs**

I. PRELIMINARIES

A) Notation and Terminology

We first present some basic notations and terminologies as used in the context of graphs and suborbital graphs that shall be used in the sequel A_n -Alternating group of degree n and order $\frac{n!}{2}$; $|G|$ -The order of a group G; $X^{(3)}$ -The set of an unordered triples from set $X=\{1,2,\ldots,n\}$; {a,b,c} -Unordered triple;

Definition 1.1

A graph G is an ordered pair (V,E), where V is a non-empty finite set of vertices and E is a set of pairs of distinct vertices in G, called edges. A loop is an edge from a vertex to itself.

Definition 1.2

A multigraph is a graph which is allowed to have multiple edges, but no loops.

Definition 1.3

If e = {u,v} is an edge of a graph G, then u and v are the end vertices of e, and we say u and v are adjacent in G. This relation is often denoted by u~v.

Definition 1.4

The degree or valency $d_G(v)$ of a vertex v of graph G is the number of edges incident to v. A vertex of degree O is an isolated vertex. Graph G is the number of edges incident to v.

Definition 1.5

A walk of length k joining u and v in G is a sequence of vertices and edges of G of the form $v_0,e_1,v_1,e_2,v_2,\text{---},v_{k-1},e_k,v_k$, where $v_0=u,v_k=v$ and $e_i=\{v_{i-1},v_i\}$ for i=1,2,---,k. A walk joining u and v is closed if u=v, and is a path if no two vertices of the walk (except possibly u and v) are equal; a closed path is called a circuit. Note that the edges $e_1,\text{---},e_k$ will frequently be omitted from the definition of a walk.

Definition 1.6

A graph G is connected if every pair of vertices of G is joined by some path; otherwise, G is disconnected.

Definition 1.7

A graph D, or a directed graph consists of a finite non empty set V=D(V) of vertices together with a collection of ordered pairs of distinct vertices of V.

Definition 1.8

Let G be transitive on X and let G_x be the stabilizer of a point $x \in X$. The orbits $\Delta_0=\{x\},\Delta_1,\Delta_2,\text{---},\Delta_{r-1}$ of G_x on X are called the suborbits of G. The rank of G is r and the sizes $n_i = |\Delta_i|$ (i=0,1,---,r-i), often called the 'lengths' of the suborbits, are known as the subdegrees of G. Note that both r and the cardinalities of the suborbits Δ_i (i=0,1,---,r-1) are independent of the choice of $x \in X$.

Definition 1.9

Let Δ be an orbit of G_x on X. Define $\Delta^*=\{gx \mid g \in G, x \in g\Delta\}$, then Δ^* is also an orbit of G_x and is called the G_x-orbit (or the G-suborbit) paired with Δ. Clearly $|\Delta|=|\Delta^*|$. If $\Delta^*=\Delta$, then Δ is called a self-paired orbit of G_x.

Theorem 1.10 [Wielandt 1964]

G_x has an orbit different from {x} and paired with itself if and only if G has even order.

Observe that G acts on X x X by g(x,y)=(gx,gy), $g \in G$, $x,y \in X$.

If $O \subseteq X \times X$ is a G-orbit, then for a fixed $x \in X$, $\Delta = \{y \in X \mid (x,y) \in O\}$ is a G_x-orbit.

Conversely if $\Delta \subseteq X$ is a G_x-orbit, then $O = \{(gx,gy) \mid g \in G, y \in \Delta\}$ is a G-orbit on X x X. We say that Δ corresponds to O. The G- orbits on X x X are called suborbitals. Let $O_i \subseteq X \times X$, i = 0,1,---,r-1 be a suborbital. Then we form a suborbital graph Γ_i, by taking X as the set of vertices of Γ_i and by including a directed edge from x to y ($x,y \in X$) if and only if $(x,y) \in O_i$. Thus each suborbital O_i determines a suborbital graph Γ_i . Now $O_i^* = \{(x,y) \mid (y,x) \in O_i\}$ is a G-orbit.

Theorem 1.11 [Sims 1967]

Let Γ_i^* be the suborbital graph corresponding to the suborbital O_i^*. Let the suborbit Δ_i (i=0,1,---,r-1) correspond to the suborbital O_i. Then Γ_i is undirected if Δ_i is self-paired and Γ_i is directed if Δ_i is not self-paired.

Theorem 1.12 [Sims 1967]

Let G be transitive on X. Then G is primitive if and only if each suborbital graph Γ_i (i=1,2,---,r-1) is connected.

Theorem 1.13 [Wielandt 1964]

Let G be transitive on X and let G_x be the stabilizer of the point $x \in X$. Let $\Delta_0 = \{x\}, \Delta_1, \Delta_2, ---, \Delta_{k-1}$ be orbits of G_x on X of lengths $n_o=1, n_1, n_2, ---, n_{k-1}$, where $n_o \le n_1 \le n_2 \le --- \le n_{k-1}$. If there exists an index j>0 such that $n_j > n_1 n_{j-1}$, then G is imprimitive on X.

II. INTRODUCTION

In 1967, Sims [6] introduced suborbital graphs corresponding to the non-trivial suborbits of a group. He called them orbitals. In1977, Neumann [4] extended the work of Higman [2] and Sims [6] to finite permutation groups, edge coloured graphs and Matrices. He constructed the famous Peterson graph as a suborbital graph corresponding to one of the nontrivial suborbits of S_5 acting on unordered pairs from the set $X=\{1,2,3,4,5\}$. The Peterson graph was first introduced by Petersen in 1898 [5]. In1992, Kamuti [3] devised a method for constructing some of the suborbital graphs of $PSL(2,q)$ and $PGL(2,q)$ acting on the cosets of their Maximal dihedral sub-groups of orders $q-1$ and $2(q-1)$ respectively. This method gave an alternative way of constructing the Coxeter graph which was first constructed by Coxeter in 1986[1]. In this paper, through computation of the rank and subdegrees of alternating group An (n=5,6,7) on unordered triples, we construct the suborbital graphs corresponding to the suborbits of these triples and further investigate properties of the suborbital graphs constructed.

A) SUBORBITAL GRAPHS OF $G=A_n$ ACTING ON $X^{(3)}$

In this section, we construct and discuss the properties of the suborbital graphs of $G=An$ acting on $X^{(3)}$.

2.1 The suborbital graphs of $G=A_5$ acting on $X^{(3)}$

The number of orbits of $G_{\{1,2,3\}}$ acting on $X^{(3)}$ is 3. These are:

$Orb_{G_{\{1,2,3\}}}\{1,2,3,\} = \{\{1,2,3\}\} = \Delta_0$, the trivial orbit.

$Orb_{G_{\{1,2,3\}}}\{1,2,4\}=\{\{1,2,4\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \{2,3,4\}, \{2,3,5\}\} = \Delta_1$, which is the set of all unordered triples containing exactly two of 1, 2 and 3.

$Orb_{G_{\{1,2,3\}}}\{1,4,5\}=\{\{1,4,5\}, \{2,4,5\}, \{3,4,5\}\}=\Delta_2$, which is the set of all unordered triples containing exactly one of 1, 2 and 3.

The suborbital graph corresponding to Δ_0 is the null graph and therefore not very interesting.

By Definition 1.1.9, Δ_1 and Δ_2 are self-paired. Hence by Theorem 1.11, their corresponding suborbital graphs Γ_1 and Γ_2 are undirected.

We construct Γ_1 and Γ_2 as follows:

Let A and B be any two distinct unordered triples from X = {1,2,3,4,5}.

(i) The suborbital O_1 corresponding to the suborbit Δ_1 is

$O_1 = \{ (g\{1,2,3\}, g\{1,2,4\}) \mid g \in G \}$.

Therefore in Γ_1, the suborbital graph corresponding to O_1, there is an edge from vertex A to B if and only if $|A \cap B| = 2$.

(ii) The suborbital O_2 corresponding to the suborbit Δ_2 is $O_2 = \{ (g\{1,2,3\}, g\{1,4,5\}) \mid g \in G \}$. Therefore in Γ_2, the suborbital graph corresponding to O_2, there is an edge from vertex A to B if and only if $|A \cap B| = 1$.

These graphs are as shown in the Fig. 1 and Fig. 2 below:

From the Fig. 1 and Fig. 2, we see that Γ_1 is regular of degree 6 and has girth 3 since there is an edge between each of the vertices {1,2,4}, {1,3,4} and {2,3,4}. On the other hand, Γ_2 is regular of degree 3 and has girth 5 since there is an edge between each of the vertices {1,2,3}, {3,4,5}, {1,2,4}, {1,3,5} and {2,4,5}. Moreover, Γ_1 and Γ_2 are connected, hence G acts primitively on $X^{(3)}$ by Theorem 1.3.

2.2 Suborbital graphs of $G = A_6$ acting on $X^{(3)}$ and their properties

The number of orbits of $G_{\{1,2,3\}}$ acting on $X^{(3)}$ is 4. These are:

$Orb_{G_{\{1,2,3\}}}\{1,2,3\} = \{\{1,2,3\}\} = \Delta_0$, the trivial orbit.

$Orb_{G_{\{1,2,3\}}}\{1,2,4\}=\{\{1,2,4\}, \{1,2,5\}, \{1,2,6\}, \{1,3,4\}, \{1,3,5\}, \{1,3,6\}, \{2,3,4\}, \{2,3,5\}, \{2,3,6\}\} = \Delta_1$, which is the set of all unordered triples containing exactly two of 1, 2 and 3.

$Orb_{G_{\{1,2,3\}}}\{1,4,5\}=\{\{1,4,5\}, \{1,4,6\}, \{1,5,6,\}, \{2,4,5\}, \{2,4,6\}, \{2,5,6\}, \{3,4,5\}, \{3,4,6\}, \{3,5,6\}\}=\Delta_2$, which is the set of all unordered triples containing exactly one of 1, 2 and 3.

$Orb_{G_{\{1,2,3\}}}\{4,5,6\}=\{\{4,5,6\}\}=\Delta_3$, which is the set of all unordered triples containing neither 1 nor 2 nor 3.

The suborbital graph corresponding to Δ_0 is the null graph and therefore not very interesting.

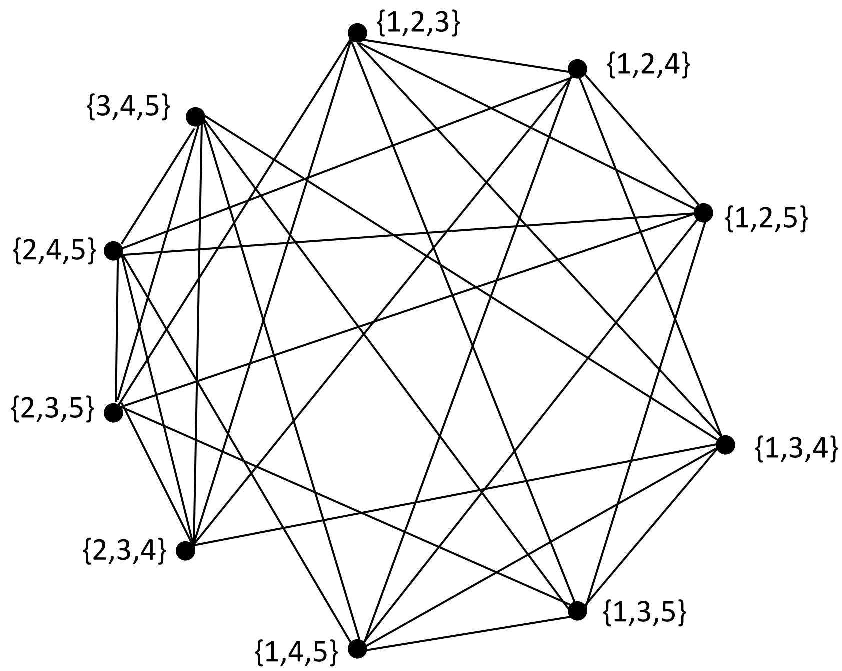

Figure 1: The suborbital graph Γ_1 corresponding to the suborbit Δ_1 of G acting on $X^{(3)}$

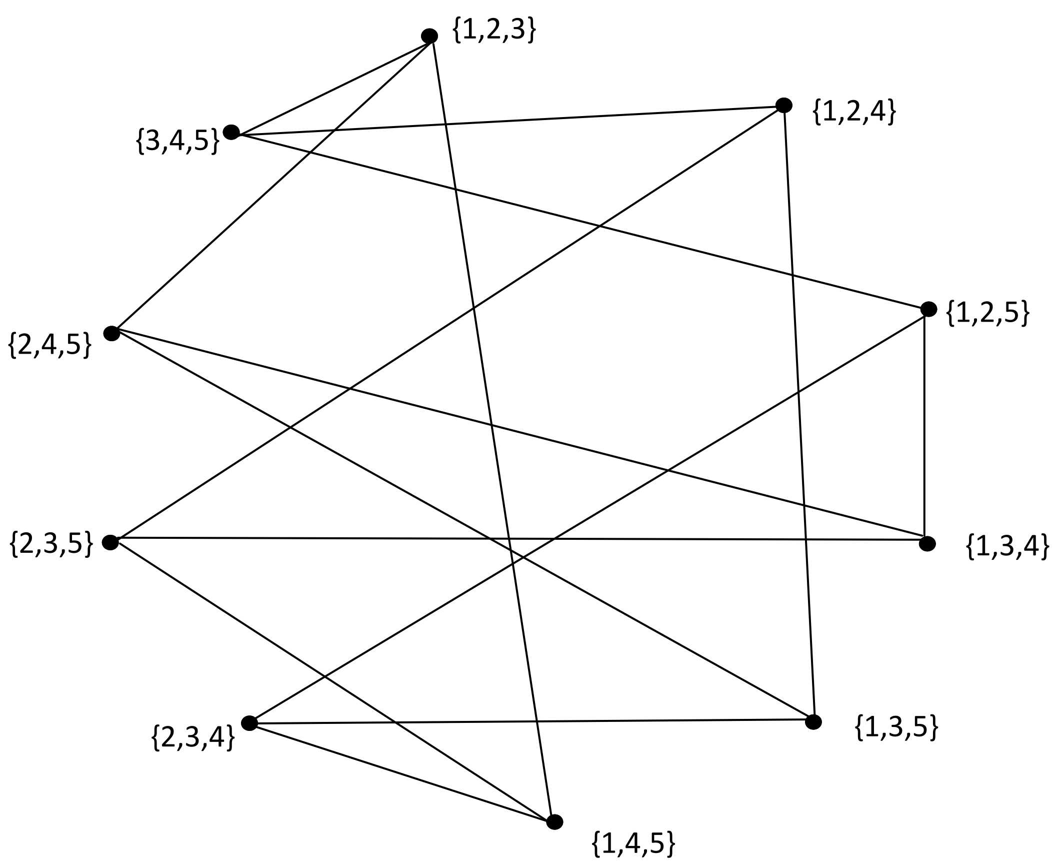

Figure 2: The suborbital graph Γ_2 corresponding to the suborbit Δ_2 of G acting on $X^{(3)}$

By Definition 1.1.9, Δ_1, Δ_2 and Δ_3 are self-paired. Hence by Theorem 1.11, their corresponding suborbital graphs are undirected.

We construct Γ_1, Γ_2 and Γ_3 as follows;

Let A and B be any two distinct unordered triples from X={1,2,3,4,5,6}.

(i) The suborbital O_1 corresponding to the suborbit Δ_1 is

$O_1 = \{ (g\{1,2,3\}, g\{1,2,4\}) \mid g \in G \}$.

Therefore in Γ_1, the suborbital graph corresponding to O_1, there is an edge from vertex A to B if and only if $|A \cap B| = 2$.

(ii) The suborbital O_2 corresponding to the suborbit Δ_2 is

$O_2 = \{ (g\{1,2,3\}, g\{1,4,5\}) \mid g \in G \}$.

Therefore in Γ_2, the suborbital graph corresponding to O_2, there is an edge from vertex A to B if and only if $|A \cap B| = 1$.

(iii) The suborbital O_3 corresponding to the suborbit Δ_3 is

$O_3 = \{ (g\{1,2,3\}, g\{4,5,6\}) \mid g \in G \}$.

Therefore in Γ_3, the suborbital graph corresponding to O_3, there is an edge from vertex A to B if and only if $|A \cap B| = 0$.

These graphs are as shown in the Fig. 3 and Fig. 4 below:

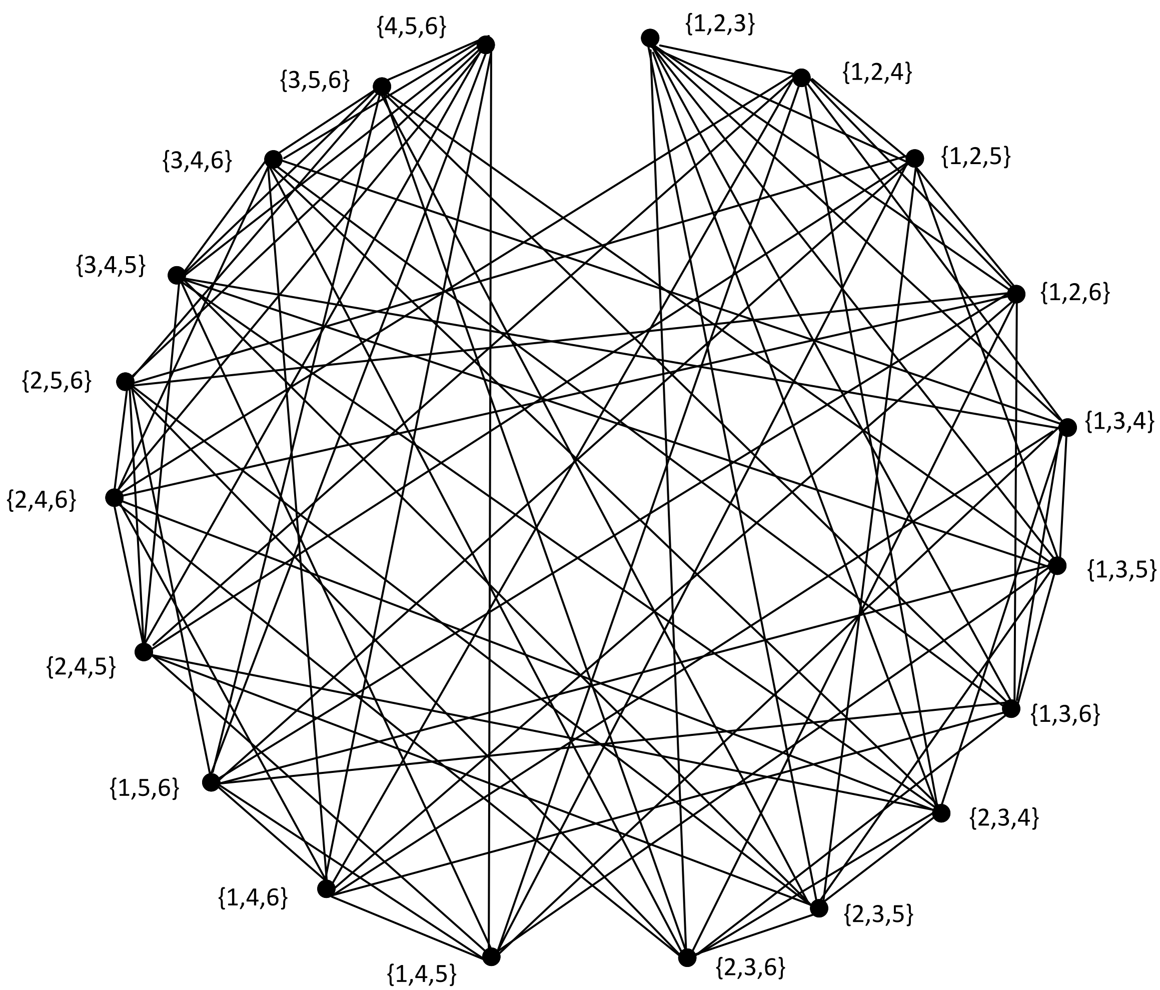

Figure 3: The suborbital graph Γ_1 corresponding to the suborbit Δ_1 of G acting on $X^{(3)}$

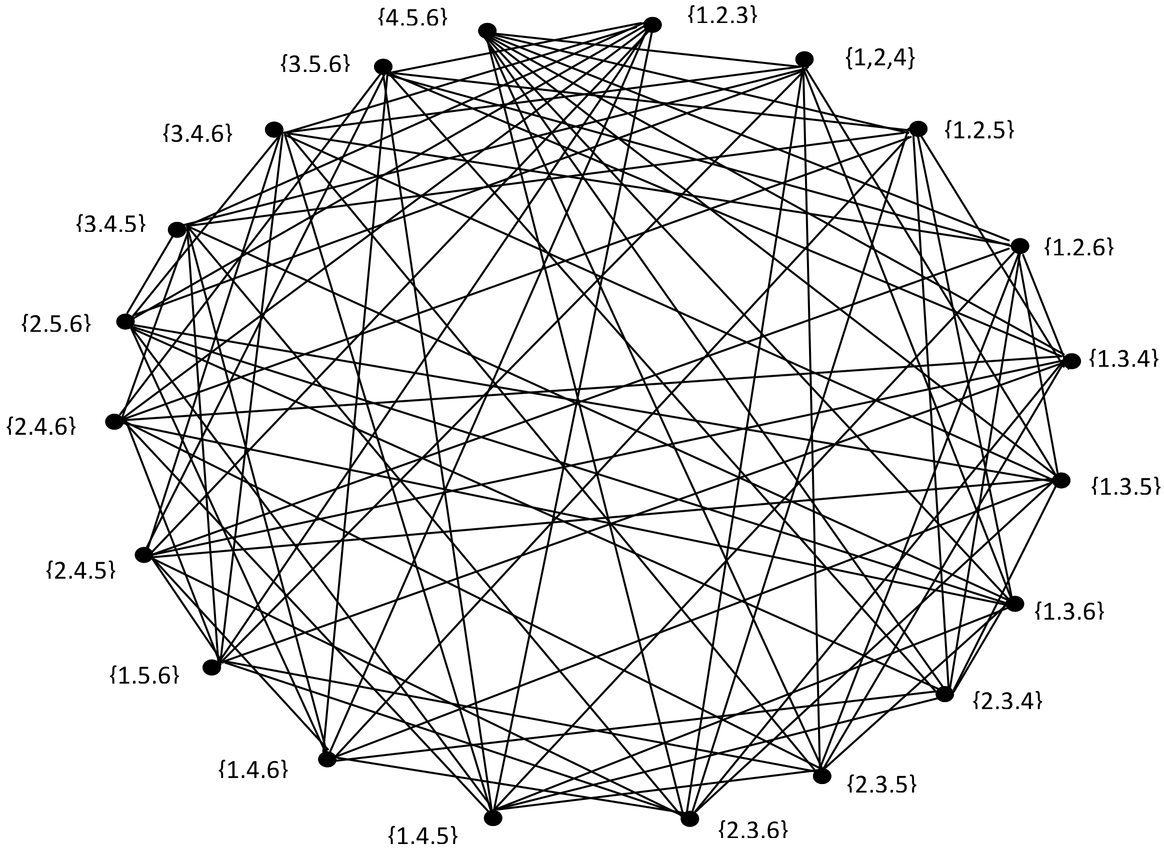

Figure 4: The suborbital graph Γ_2 corresponding to the suborbit Δ_2 of G acting on $X^{(3)}$

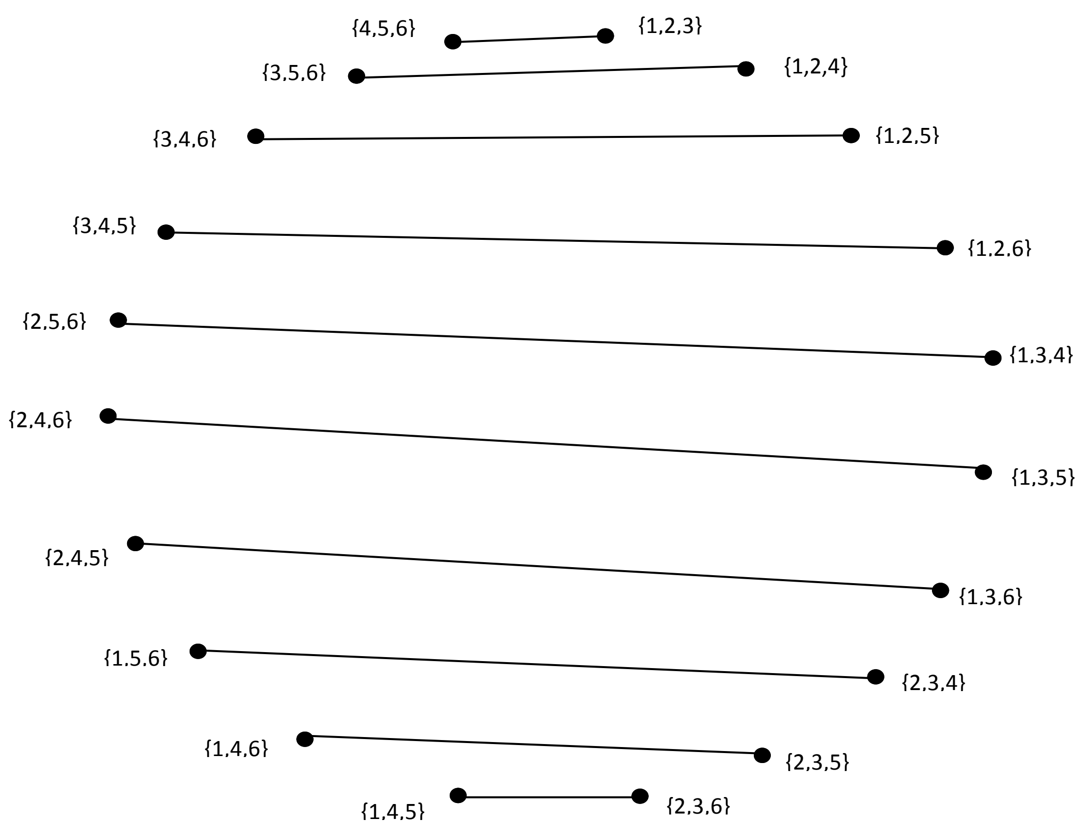

Figure 5: The suborbital graph Γ_3 corresponding to the suborbit Δ_3 of G acting on $X^{(3)}$

From the diagrams, we see that Γ_1 is connected, regular of degree 9 and has girth 3 since there is an edge between each of the vertices {1,2,3}, {1,2,4} and {1,2,5}. The suborbital graph Γ_2 is connected, regular of degree 9 and has girth 3 since there is an edge between each of the vertices {1,2,3}, {1,4,5} and {2,4,6} while Γ_3 is disconnected, regular of degree 1 and has no cycles.

2.3 Suborbital graphs of $G = A_7$ acting on $X^{(3)}$ and their properties

The number of orbits of $G_{\{1,2,3\}}$ acting on $X^{(3)}$ is 4. These are:

$Orb_{G_{\{1,2,3\}}}\{1,2,3,\} = \{\{1,2,3\}\} = \Delta_0$, the trivial orbit.

$Orb_{G_{\{1,2,3\}}}\{1,2,4\}$={{1,2,4}, {1,2,5}, {1,2,6}, {1,2,7}, {1,3,4}, {1,3,5}, {1,3,6}, {1,3,7}, {2,3,4}, {2,3,5}, {2,3,6}, {2,3,7}} =Δ_1, which is the set of all unordered triples containing exactly two of 1, 2 and 3.

$Orb_{G_{\{1,2,3\}}}\{1,4,5\}$={{1,4,5}, {1,4,6}, {1,4,7}, {1,5,6,}, {1,5,7}, {1,6,7}, {2,4,5}, {2,4,6}, {2,4,7}, {2,5,6}, {2,5,7}, {2,6,7}, {3,4,5}, {3,4,6}, {3,4,7}, {3,5,6}, {3,5,7}, {3,6,7}}=Δ_2, which is the set of all unordered triples containing exactly one of 1, 2 and 3.

$Orb_{G_{\{1,2,3\}}}\{4,5,6\}$={{4,5,6}, {4,5,7}, {4,6,7}, {5,6,7}}=Δ_3 which is the set of all unordered triples containing neither 1 nor 2 nor 3.

The suborbital graph corresponding to Δ_0 is the null graph and therefore not very interesting.

By definition 1.9, Δ_1, Δ_2 and Δ_3 are self-paired. Hence by Theorem 1.11, their corresponding suborbital graphs are undirected.

We construct Γ_1, Γ_2 and Γ_3 as follows:

Let A and B be any two distinct unordered triples from X={1,2,3,4,5,6,7}.

(i) The suborbital O_1 corresponding to the suborbit Δ_1 is

$O_1 = \{ (g\{1,2,3\}, g\{1,2,4\}) \mid g \in G \}$

Therefore in Γ_1, the suborbital graph corresponding to O_1, there is an edge from vertex A to B if and only if $|A \cap B| = 2$.

(ii) The suborbital O_2 corresponding to the suborbit Δ_2 is

$O_2 = \{ (g\{1,2,3\}, g\{1,4,5\}) \mid g \in G \}$.

Therefore in Γ_2, the suborbital graph corresponding to O_2, there is an edge from vertex A to B if and only if $|A \cap B| = 1$.

(iii) The suborbital O_3 corresponding to the suborbit Δ_3 is

$O_3 = \{ (g\{1,2,3\}, g\{4,5,6\}) \mid g \in G \}$.

Therefore in Γ_3, the suborbital graph corresponding to O_3, there is an edge from vertex A to B if and only if $|A \cap B| = 0$.

These graphs are as shown in the Fig. 6, Fig. 7 and Fig. 8 below:

From the diagrams, we see that Γ_1 is regular of degree 12 and has girth 3 since there is an edge between each of the vertices {5,6,7}, {3,5,7} and {2,5,7}. The suborbital graph Γ_2 is regular of degree 18 and has girth 3 since there is an edge between each of the vertices {1,2,3}, {1,5,6} and {2,4,6} while Γ_3 is regular of degree 4 and has girth 7. Moreover Γ_1, Γ_2 and Γ_3 are connected hence G acts primitively on $X^{(3)}$.

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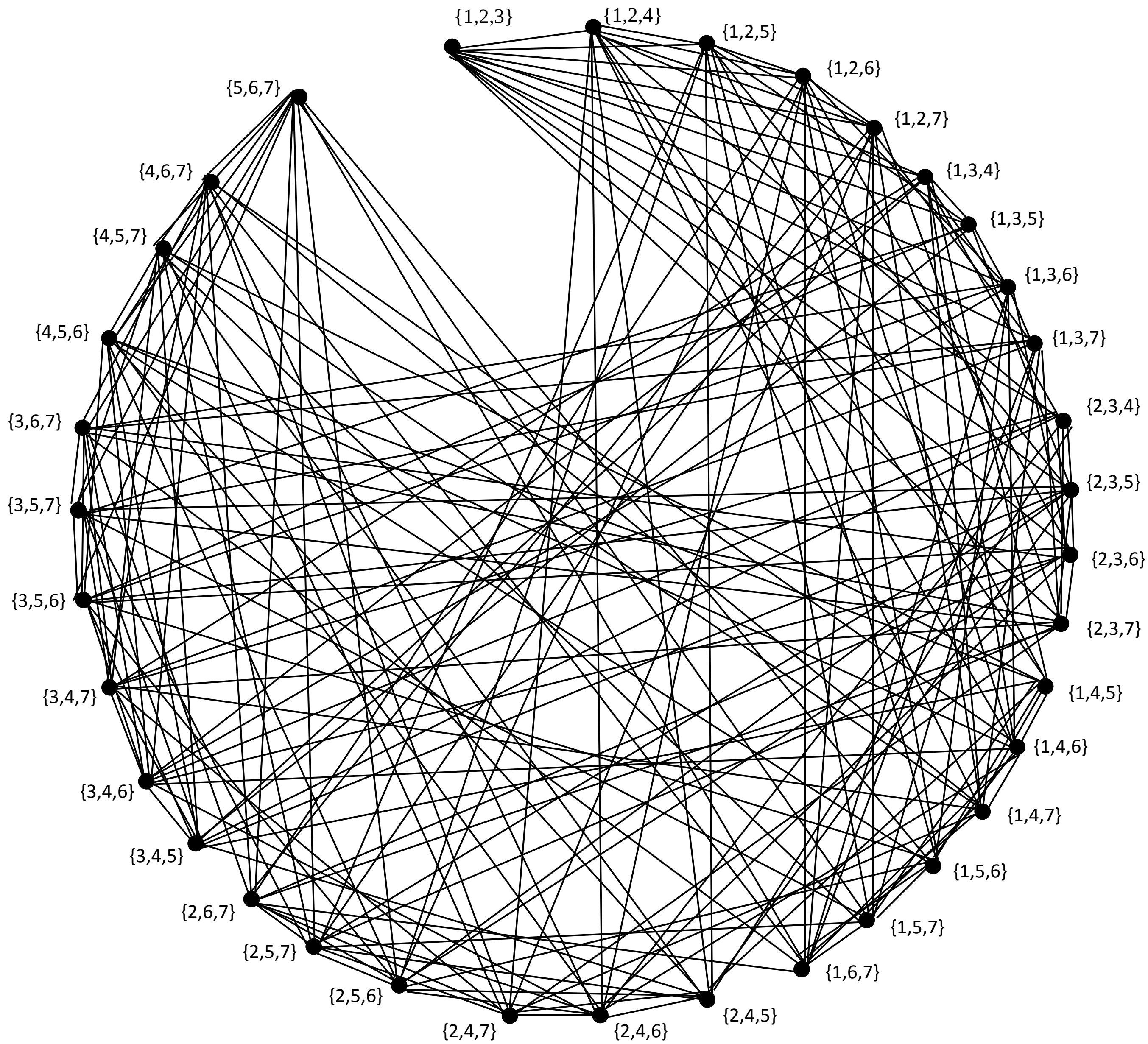

Figure 6: The suborbital graph Γ_1 corresponding to the suborbit Δ_1 of G acting on $X^{(3)}$

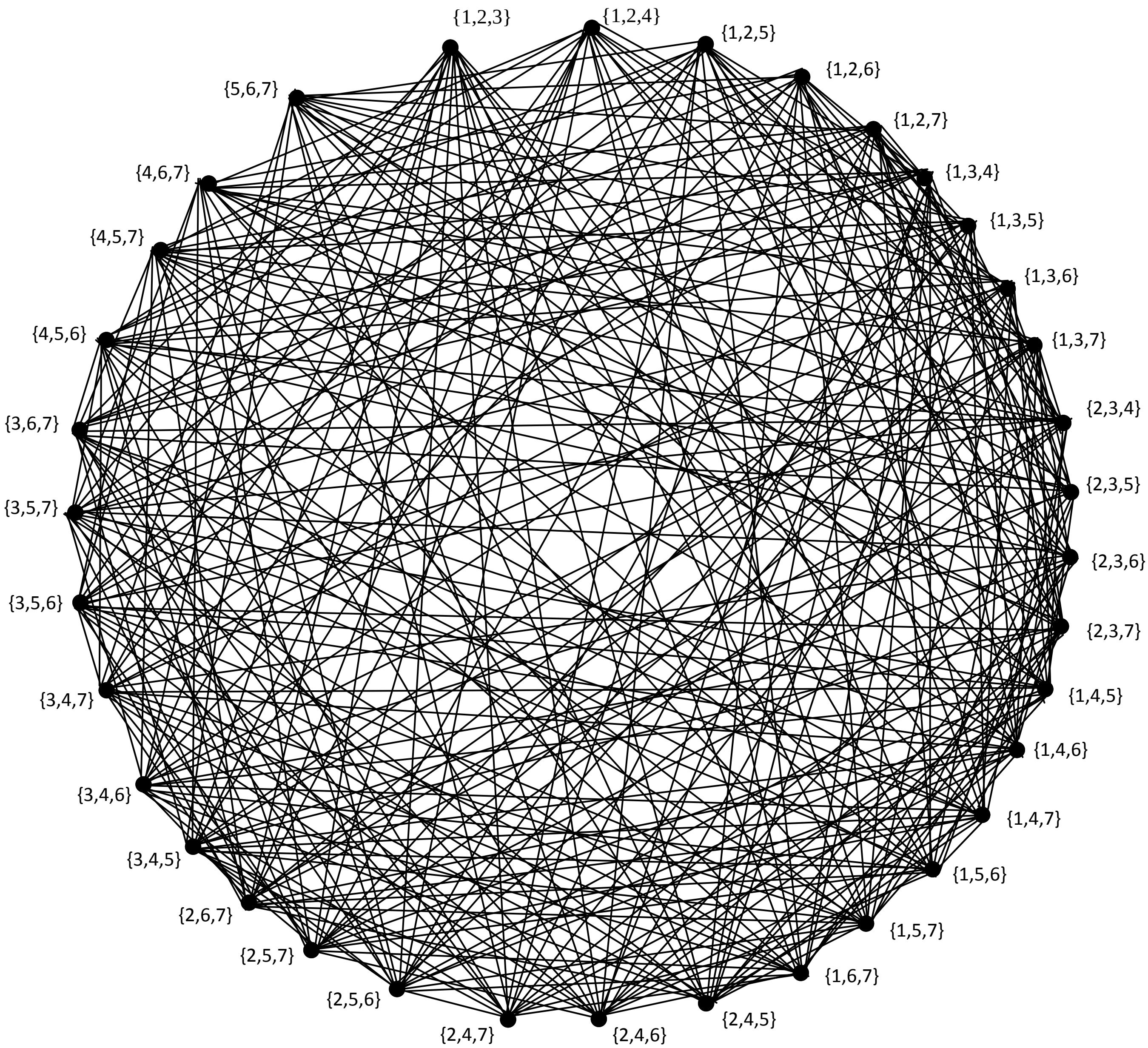

Figure 7: The suborbital graph Γ_2 corresponding to the suborbit Δ_2 of G acting on $X^{(3)}$

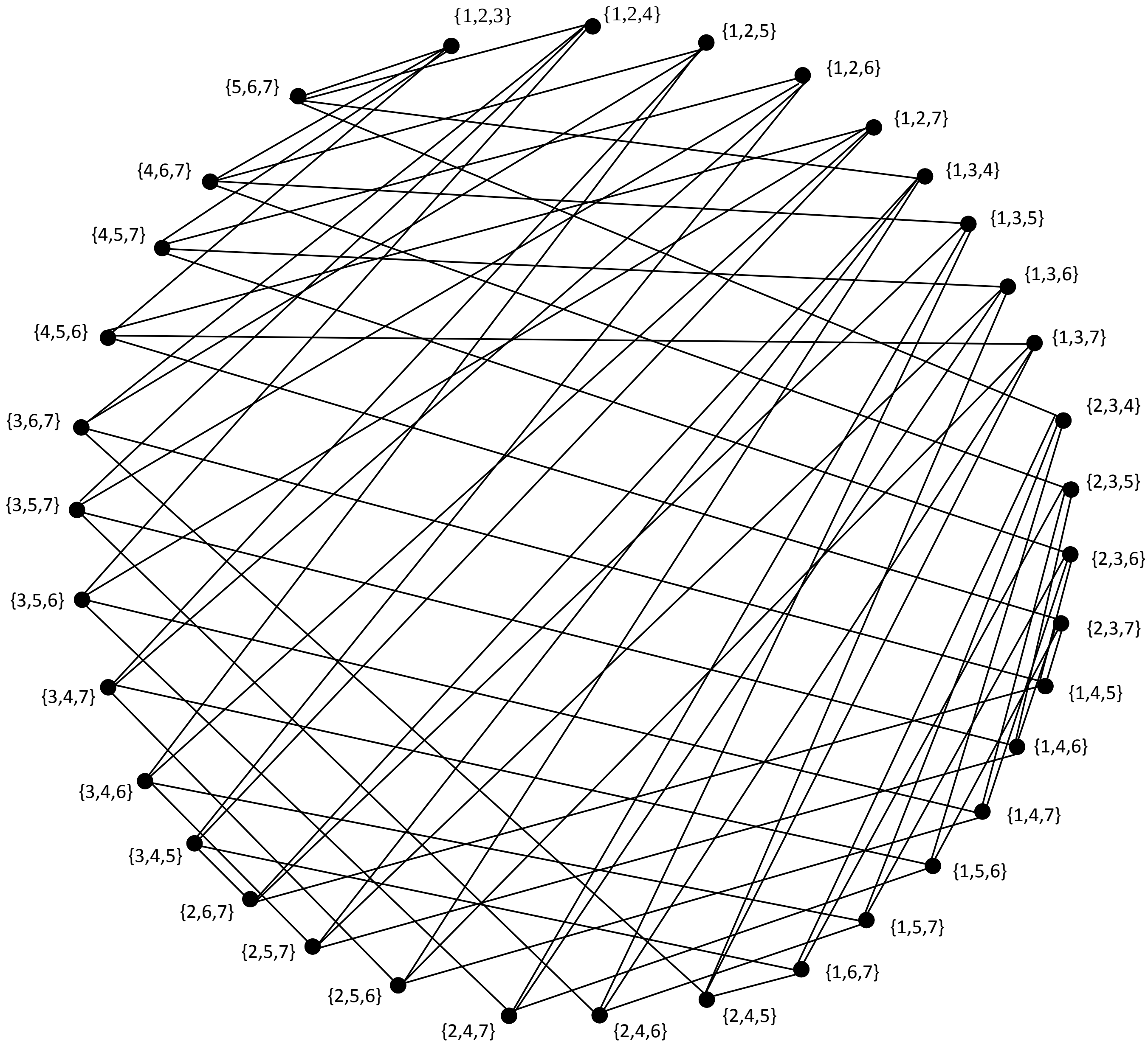

Figure 8: The suborbital graph Γ_3 corresponding to the suborbit Δ_3 of G acting on $X^{(3)}$