Comparison Solutions of the Laplace's Equation by the FDM and BEM with Simple Boundary Conditions

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Abstract— **This paper presents to solve the Laplace's equation that is many applicable to the engineering problems. Analytical solution may be calculated by separation of variables method. Numerically solutions by two methods i.e. the finite difference method (FDM) and the boundary element method (BEM) are employed to predict the potential characteristics of the domain. In the BEM, the integration domain needs to be discretized into small elements. The boundary integral equation derived using Green's theorem by applying Green's identity for any point in the surface. The methods are applied to investigate an example of rectangular domain. The present methods can be extrapolated to other linear homogeneous differential equations. Both types of numerical models are computed and compared with analytical solution. The results obtained agree perfectly with those obtained from exact solution.**

Keywords— **Laplace's equation, Boundary Element Method, and Finite Difference Method**

I. INTRODUCTION

In mathematics, a PDE is a differential equations involving functions and their partial derivatives. PDEs are used to formulate problems involving functions of several variables, and are either solved by hand, or used to create a relevant computer model. PDEs can be used to describe a wide variety of phenomena such as sound, heat, electrostatics, electrodynamics, fluid flow, elasticity, or quantum mechanics. These seemingly distinct physical phenomena can be formalized similarly in terms of PDEs. Just as ordinary differential equations often model one-dimensional dynamical systems, partial differential equations often model multi-dimensional systems.

Because of its vast range of application at all science branches, lots of scientists and engineers worked on this equation. In the literature, Carrier and Pearson [1], Muskhelishvili [2], Ling [3], Timoshenko and Goodier [4], Lebedev [5], Gilbarg and Trudinger [6] have solved this kind of problems. In the sample of earlier works, Yu Xie Mukherjee, and Subrata Mukherjee studied on the boundary node method for potential problems using various boundary conditions of the Dirichlet, Neumann and mixed problems [7]. Qian et al solved a Cauchy problem for the Laplace equation in a rectangle [8]. They proposed two different regularization methods on the ill-posed problem based on separation of variables. Jain et al proposed exact numerical closed-form expressions for potential coefficient [9]. The basic idea was to express all potential coefficients of suitably defined virtual plates. Furthermore, the Laplace equation has to be solved numerically. Chen and Shen solved a semi-analytical method for the Laplace problems with circular boundaries. They used the degenerate kernels to avoid calculating the principal values [10]. Their achievement are five advantages, well-posed linear algebraic system, principal value free, elimination of boundary-layer effect, exponential convergence and mesh free. Chen et al focused on the connection between conformal mapping and curvilinear coordinates and figure out the relation to take integration by way of mapping in the complex plane [11]. Chen et al used the image method to solve boundary value problems in domain containing circular and spherical shaped. Two and three dimensional problems as well as symmetric and anti-symmetric cases are considered [12]. Same authors derived the Green's function using the bipolar coordinates, image method and the method of fundamental solutions of Laplace problems containing circular boundaries [13]. They studied on the optimal locations of sources in the method of fundamental solutions that are dependent on the source location and the geometry of problems. Matthew et al applied the explicit FDM for solving singularity problems on potential computation in spheroidal systems [14]. They carried out method for treating the singularities at the pole regions which impose a Neumann boundary condition along the two lines of symmetry. Morales et al [15] studied on the solutions of Laplace's equation with simple boundary conditions, with consideration to their applications for capacitors with multiple symmetries.

The objective of this work is to determine potential in rectangular domain using BEM and FDM. The numerical results and analytical solution are presented here. Comparisons reveal that the methods are efficient and the results are in good agreement with analytical measurements.

The paper is arranged hereafter: In section II, we solve Laplace's equation in simple geometry by separation of variables. As a matter of illustration of the method, we obtain the potential quantity for this case. In section III, we applied BEM in order to obtain potential at each point of domain. On the other hand, we use FDM in Section IV to determine the considerable value. Section V shows results of two numerical models, BEM and FDM. Also we compare the results with analytical data. Finally, Section VI contains our conclusions.

II. EXACT SOLUTION

The Laplace equation and its boundary conditions (Fig. 1) are defined as follows:

$$\nabla^2 \varphi = 0 \qquad (1)$$
$$BC : \varphi = \bar{u}$$

Separation of variables is any of several methods for solving ordinary and partial differential equations, in which algebra allows one to rewrite an equation so that each of two variables occurs on a different side of the equation. It works because it reduces a PDE to ODEs. All the steps of solution can be expressed as follows. In this method we attempt to determine solutions in the product form. Next we place this in Laplace's equation. Then we claim it is necessary that both sides of the equation must equal to same constant known as the separation constant named Lambda. Now we should do this for any arbitrary constant due to boundary condition. However eventually we will discover that only certain values of Lambda are allowable. Now we obtained the Eigen functions of each variable. The original $\varphi(x,y)$ is obtained by multiplying together the variables. In summary, we obtained product solutions of the Laplace's equation satisfying the specific homogenous boundary conditions only corresponding to $\lambda > 0$. These solutions, $\varphi(x,y) = X(x)Y(y)$, have $X(x) = C_1 sin\sqrt{\lambda}x$ and $Y(y) = C_2 sinh\sqrt{\lambda}y$, where we determined from the boundary conditions the allowable values of the separation constant λ, $\lambda = (n\pi)^2$. Here n is a positive integer.

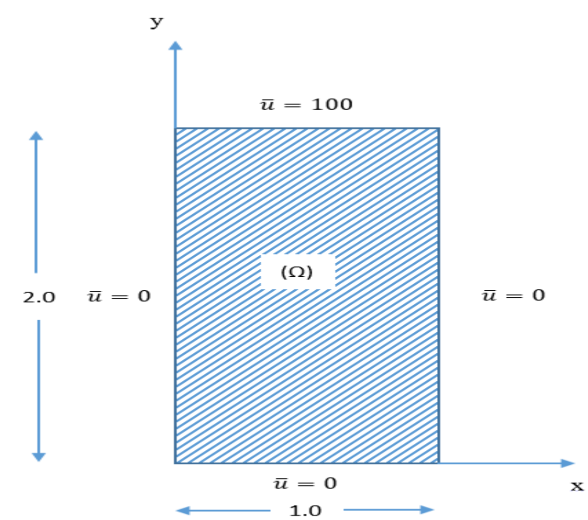

Fig. 1. Boundary condition

Thus, product solutions of the Laplace's equation are

$$\varphi(x,y) = \sum_{n=1}^{\infty} A_n \sin n\pi x \sinh n\pi y \qquad (2)$$

where: $A_n = 200\int_0^1 \sinh 2n\pi \sin n\pi x dx$

So exact solution of equation is obtained as follows

$$\varphi(x,y) = \sum_{n=1}^{\infty} \frac{200}{n\pi \sinh 2n\pi}\left[1-(-1)^n\right] \sin n\pi x \sinh n\pi y \qquad (3)$$

III. BOUNDARY ELEMENT METHOD (BEM)

In the BEM, Laplace's equation may be transformed into integral form based on Green's theorem. This is a numerical method for solving partial differential equations encountered in mathematical physics and engineering. The integrals in the BEM are numerically integrated over the boundary. The boundary is divided into small elements, as well as other numerical methods ultimately a linear algebraic equation will be obtained which has only one answer. BEM is simply and geometrically applied for any complex shape. In addition, BEM can model areas with sharp changes in variables with accuracy better than the FDM because all approximations limited to the surface. The surface is divided into sections and elements. Shape functions are used to describe the variables and geometries for each element. These shape functions can be linear, quadratic and higher orders. In this way due to the complexity of integrating functions, the analytical integration is not recommended to calculate integrals and numerical integration and Gaussian square method is used instead of it. When the points are close to each other in the calculation of singular integrals or source point p matches the boundary point q, special relationships must be used. Because of the main solution contains orders of 1/r. The total integral is calculated by adding all the integrals on all elements our approach to solve this equation is that the use of fixed element according to the geometry and boundary conditions.

In the BEM for a body of the boundary and domain, the integral formulation may be expressed as:

$$e(p)\varphi(p) = \int_{\Gamma}\left[\varphi\frac{\partial G}{\partial n} - G\frac{\partial \varphi}{\partial n}\right]ds \qquad (4)$$

where:

$$e(p) = \begin{cases} 1 \ \textit{for P inside domain} \\ 0.5 \ \textit{for P on surface} \\ 0 \ \textit{for P outside domain} \end{cases} \qquad (5)$$

And G is the green's function of the Laplace equation.

$$G = \frac{1}{2\pi} Lnr \qquad (6)$$

where r distance between the source point and integral element expressed as

$$\vec{r} = \vec{p} - \vec{q} \qquad (7)$$

Discretization form of the equation can be represented as follows:

$$e_i\varphi_i = \sum_{j=1}^{N} L_{ij}\varphi_{n_j} - \sum_{j=1}^{N} \hat{H}_{ij}\varphi_j \qquad (8)$$

where L_{lj} and H_{lj} are influence coefficients and defined as follows:

$$L_{lj} = \int_{\Gamma} G dS \tag{9}$$

$$\hat{H}_{lj} = \int_{\Gamma} G_n dS \tag{10}$$

The integral can be evaluated by numerical and analytical methods. Moreover setting

$$H_{lj} = 0.5\,\delta_{lj} + \hat{H}_{lj} \tag{11}$$

where δ_{lj} is the Kronecker delta, which is defined as

$$\delta_{lj} = \begin{cases} 0 & for\ l \neq j \\ 1 & for\ l = j \end{cases} \tag{12}$$

The term of the RHS of the (8) can be calculated by the integral on the boundary. If the Dirichlet boundary condition employs all terms of the u will be completely known. Then (8) can be arranged as

$$\sum_{j=1}^{N} [H_{lj}]\{\varphi_j\} = \sum_{j=1}^{N} [L_{lj}]\{\varphi_{n_j}\} \tag{13}$$

Equation (13) may be solved and the unknown variables are obtained.

At this stage, the values obtained on all boundaries now it must be assumed as a source to resolve in internal point and its relationship. With all the existing elements is obtained as follows:

$$\varphi(p) = \int \varphi G_n ds - G\varphi_n ds \tag{14}$$

where the matrix form is expressed as

$$\varphi(p) = [\hat{H}_{p1}\hat{H}_{p2}\cdots\hat{H}_{p24}]\begin{Bmatrix}\varphi_1\\ \varphi_2\\ \vdots\\ \varphi_{24}\end{Bmatrix} - [\hat{G}_{p1}\hat{G}_{p2}\cdots\hat{G}_{p24}]\begin{Bmatrix}\varphi_{n_1}\\ \varphi_{n_2}\\ \vdots\\ \varphi_{n_{24}}\end{Bmatrix} \tag{15}$$

First, the point p is placed in the midpoint of element 1 that provides a set of equations. This collection is related to the all N variables to each other on the surface. In the following point p is placed in the middle of element 2 and leads to a series of equations at the point. Note that the equation should be organized so that all the unknowns placed on the left side of the equation and the known values placed on the right. A direct solution method such as Gaussian methods is used because the final matrix is non-symmetric and full of nonzero coefficients.

Furthermore, in the BEM, we solve only for the boundary distribution of the unknown function or one of its derivatives. It is not necessary to compute the requisite function throughout the domain of solution. Once the unknown boundary distribution is available, the solution at any point may be produced by direct evaluation. Thus, the crux of the BEM is the reduction of the dimension of the solution space with respect to physical space by one unit.

Despot of, some initial effort is required to learn the fundamental principles underlying the integral representations and the implementation of the numerical methods. An oversight in the mathematical formulation, a mistake in the implementation of the numerical methods, or an error in a code is likely to have a catastrophic effect on the accuracy of the solution.

IV. FINITE DIFFERENCE METHOD (FDM)

FDM is one of the easiest and an oldest numerical method because of its simplicity is commonly used by engineers. However, due to the inability of this method in spatial discretization of non-rectangular and complex geometries, its usage is limited to relatively simple and rectangular geometry issues. In FDM Taylor series expansion and equations such as these are used. Derivatives and equations approximate and directly replace with various terms in the equations. Finite difference estimates are discrete model of continuous finite difference operators. They are used to provide a discrete model of a partial differential equation. Finite differences associated with the derivative of a function estimate at a point such as X0 obtained by using the function values in the vicinity of the point. These estimates have been usually formed of function values in a certain number of points that have been placed at the same distance apart. Estimates of finite difference can be divided to smaller categories of backward, forward and central. For this article according to the Laplace's equation central three-point discretization for second order derivatives were used in the following equations:

$$\frac{\partial^2\varphi}{\partial x^2}|(x_0,y_0) = \frac{1}{(\Delta x)^2}\left[\varphi(x_0-\Delta x, y_0) - 2\varphi(x_0,y_0) + \varphi(x_0+\Delta x, y_0)\right] \tag{16}$$

$$\frac{\partial^2\varphi}{\partial y^2}|(x_0,y_0) = \frac{1}{(\Delta y)^2}\left[\varphi(x_0, y_0-\Delta y) - 2\varphi(x_0,y_0) + \varphi(x_0, y_0+\Delta y)\right] \tag{17}$$

where $\Delta x = \Delta y = h = 0.25$

By substituting x in Laplace's equation, discretized equation can be obtained as follows:

$$\varphi(x_0-h, y_0) + \varphi(x_0, y_0-h) - 4\varphi(x_0,y_0) + \varphi(x_0+h, y_0) + \varphi(x_0, y_0+h) = 0 \tag{18}$$

After discretization, boundary conditions must be applied. The boundary condition on Dirichlet type is applied directly in the discrete equation. Boundary derivatives can be approximated with finite difference and placed it in the system of linear equations. One thing that is important is that truncation error finite difference estimations for boundary derivative should be equal to finite difference truncation error for differential equation, because of its high accuracy in validation. Fig. 2 shows the discretizations of the elements in the BEM and FDM.

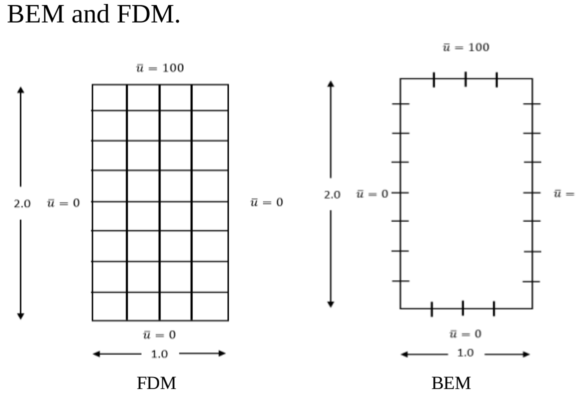

Fig. 2. Boundary discretization on both methods

V. RESULTS

In the BEM, number of the boundary element are 24 while the elements number of the FDM is 32. Comparison of the potential by the numerical and analytical results of the potential is shown through Tables 1 and 2 when the field point is located on the middle of each element. The approach of this paper is based on considering equal elements on boundary for BEM and FDM to compare these numerical methods with each other in same level of accuracy. Generally, the error of the BEM is less than the FDM in all points. These solutions are obtained when number of elements in the BEM is less than the FDM. If we increase the number of the elements in the BEM the relative error may diminishes much more than the present results.

Table 1. Comparison of the potential (analytical and BEM)

0.25	0.25	0.292012	0.28950	0.008602
0.5	0.25	0.41309	0.4099	0.007722
0.75	0.25	0.292012	0.28950	0.008602
0.25	0.5	0.773854	0.76 58	0.010407
0.75	0.5	0.773854	0.7658	0.010407
0.25	0.75	1.7581182	1.7417	0.009374
0.75	0.75	1.7581182	1.7417	0.009374
0.25	1	3.88579	3.8551	0.007898
0.75	1	3.88579	3.8551	0.007898
0.25	1.25	8.55535	8.5002	0.006231
0.75	1.25	8.55535	8.5002	0.006231
0.25	1.5	18.9767	18.8624	0.006023
0.75	1.5	18.9767	18.8624	0.006023
0.25	1.75	43.49488	43.4986	0.000855
0.5	1.75	54.466	54.6388	0.003172
0.75	1.75	43.49488	43.4986	0.000855

Table 2. Comparison of the potential (analytical and FDM)

0.25	0.25	0.292012	0.353	0.208854
0.5	0.25	0.41309	0.4989	0.207730
0.75	0.25	0.292012	0.353	0.208854
0.25	0.5	0.773854	0.9132	0.180071
0.75	0.5	0.773854	0.9132	0.180071
0.25	0.75	1.7581182	2.0103	0.143397
0.75	0.75	1.7581182	2.0103	0.143397
0.25	1	3.88579	4.2957	0.105490
0.75	1	3.88579	4.2957	0.105490
0.25	1.25	8.55535	9.1532	0.070112
0.75	1.25	8.55535	9.1532	0.070112
0.25	1.5	18.9767	19.6632	0.036176
0.75	1.5	18.9767	19.6632	0.036176
0.25	1.75	43.49488	43.2101	0.0065474
0.5	1.75	54.466	53.1774	0.0236589
0.75	1.75	43.49488	43.2101	0.0065474

VI. CONCLUSIONS

In this paper, a two dimensional numerical model using the BEM was introduced to predict the potential characteristics of various point of domain. In addition, the FDM applied to compare with. The numerical results of the potential distributions were compared against exact data and shown to be a good agreement. As a result of the present work, the following conclusions can be drawn:

- A comparison of the potential distributions for the various points of geometry shows satisfactory results compared with those exact and numerical data.
- Alternative methods require discretizing the whole of the solution domain, and this considerably raises the cost of the computation.
- Problems that can be solved on a laptop computer using the BEM may require the use of a supercomputer by FDM for the same level of accuracy.

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