Lie Symmetry Solution of Fourth Order Ordinary Differential Equation: $y^{(4)} = (y)^{-1} y'y'''$

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***Abstract*— Lie group theory is a mathematical tool used in solving nonlinear differential equations. These differential equations have turned up to be very handy in applied mathematics. This theory is applied to differential equations occurring as mathematical models. It's rich, very interesting and in broad sense a topic of active mathematical research. Efforts to obtain solutions to a variety of differential equations by use of Lie symmetry have been in existence for a long period of time. In this paper we solve a harmonic fourth order non-linear ordinary differential equation $y^{(4)} = (y)^{-1} y'y'''$, and obtain its general solution using Lie Symmetry Group Invariant method. This method makes use of Infinitesimal transformations, symmetries and integrating factors.**

***Keywords*— Differential Equation, Generator, Infinitesimal Transformation and Lie Group**

I. INTRODUCTION

Lie groups of transformations are characterized by infinitesimal generators and the application of the same only requires knowledge of the admitted infinitesimal generators. A lie group of transformation admitted by a differential equation corresponds to a mapping of each of its solutions to another solution of the same differential equation [5]. Many differential equations of practical interest evolve on Lie groups or on manifolds acted upon by Lie groups [1].

Differential equations are vitally important in numerous scientific fields. Oftentimes, they are quite challenging to solve. Differential equations are used to model numerous phenomenons in our world, from the spread of infectious diseases to the behavior of tidal waves [3]. Naturally, the study of differential equations plays a vital role in the physical sciences. These equations are often non-linear and solving them requires unique and creative methods.

II. OBTAINING THE GENERAL SOLUTION FOR THE NON-LINEAR DIFFERENTIAL EQUATION $y^{(4)} = (y)^{-1} y'y'''$

Let us consider the differential equation:

$$y^{(4)} = (y)^{-1} y'y''' \tag{1}$$

Since our equation is of fourth order, we will subject it to the fourth extension of our generator, which takes the form [5]:

$$T^{[4]} = \xi\frac{\partial}{\partial x} + \lambda\frac{\partial}{\partial y} + (\lambda' - \xi' y')\frac{\partial}{\partial y'} + (\lambda'' - 2y''\xi' - y'\xi'')\frac{\partial}{\partial y''} + (\lambda''' - 3y'''\xi' - 3y''\xi'' - y'\xi''')\frac{\partial}{\partial y'''} + \left(\lambda^{(4)} - 4y^{(4)}\xi' - 6y'''\xi'' - 4y''\xi''' - y'\xi^{(4)}\right)\frac{\partial}{\partial y^{(4)}} \tag{2}$$

Subjecting our equation (1) to the fourth extension of the generator we obtain:

$$T^{[4]}\left[y^{(4)} - (y)^{-1} y'y'''\right] = 0 \tag{3}$$

Solving (3) yields

$$\xi\left[y^{(5)} - y^{-1}y'y^{(4)} - y^{-1}y''y''' + y^{-2}(y')^2 y'''\right] + \lambda\left[(y)^{-2} y'y'''\right] + (\lambda' - \xi' y')(-y^{-1}y''') + [\lambda'' - 2y''\xi' - y'\xi''](0) + \left[\lambda^{(4)} - 4y^{(4)}\xi' - 6y'''\xi'' - 4y''\xi''' - y'\xi^{(4)}\right](1) = 0 \tag{4}$$

From equation (1) we can easily show that

$$y^{(5)} = y^{-1}y'y^{(4)} + y^{-1}y''y''' - y^{-2}(y')^2 y''' \tag{5}$$

Substituting values of equation (5) into equation (4) we get:

$$\xi y^{-1}y'y^{(4)} + \xi y^{-1}y''y''' - \xi y^{-2}(y')^2 y''' - \xi y^{-1}y'y^{(4)} - \xi y^{-1}y''y''' + \xi y^{-2}(y')^2 y''' + \lambda y^{-2}y'y''' - \lambda' y^{-1}y''' + \xi' y^{-1}y'y''' - \lambda''' y^{-1}y' + 3\xi' y^{-1}y'y''' + 3y^{-1}y'y''\xi'' + \xi''' y^{-1}(y')^2 + \lambda^{(4)} - 4y^{(4)}\xi' - 6y'''\xi'' - 4y''\xi''' - y'\xi^{(4)} = 0 \tag{6}$$

Replacing $y^{(4)} = (y)^{-1}y'y'''$ into (6) we get:

$$\xi y^{-1}y'(y^{-1}y'y''') + \xi y^{-1}y''y''' - \xi y^{-2}(y')^2 y'''$$
$$-\xi y^{-1}y'\left(y^{-1}y'y'''\right) - \xi y^{-1}y''y''' + \xi y^{-2}(y')^2 y'''$$
$$+\lambda y^{-2}y'y''' - \lambda' y^{-1}y''' + \xi' y^{-1}y'y''' - \lambda''' y^{-1}y'$$
$$+3\xi' y^{-1}y'y''' + 3\xi'' y^{-1}y'y'' + \xi''' y^{-1}(y')^2 + \lambda^{(4)}$$
$$-4\left(y^{-1}y'y'''\right) - 6\xi'' y''' - 4\xi''' y'' - \xi^{(4)}y' = 0 \qquad (7)$$

Simplifying (7) gives us:

$$\lambda y^{-2}y'y''' - \lambda' y^{-1}y''' + \xi' y^{-1}y'y''' - \lambda''' y^{-1}y'$$
$$+3\xi' y^{-1}y'y''' + 3\xi'' y^{-1}y'y'' + \xi''' y^{-1}(y')^2 + \lambda^{(4)}$$
$$-4\xi' y^{-1}y'y''' - 6\xi'' y''' - 4\xi''' y'' - \xi^{(4)}y' = 0 \qquad (8)$$

The primes in (8) above represent derivatives. The derivatives of ξ and λ in terms of partial derivatives are used [2]. Substituting values of ξ and λ and their derivatives into equation (8) above we obtain:

$$\lambda y^{-2}y'y''' - y^{-1}y'''\frac{\partial\lambda}{\partial x} - y^{-1}y'y'''\frac{\partial\lambda}{\partial y} + y^{-1}y'y'''\frac{\partial\xi}{\partial x}$$
$$+y^{-1}y'^2y'''\frac{\partial\xi}{\partial y} - y^{-1}y'\frac{\partial^3\lambda}{\partial x^3} - 3y^{-1}y'^2\frac{\partial^3\lambda}{\partial x^2\partial y}$$
$$-3y^{-1}y'y''\frac{\partial^2\lambda}{\partial x\partial y} - y^{-1}y'y'''\frac{\partial\lambda}{\partial y} - 3y^{-1}y'^3\frac{\partial^3\lambda}{\partial x\partial y^2}$$
$$-3y^{-1}y'^2y''\frac{\partial^2\lambda}{\partial y^2} - y^{-1}y'^4\frac{\partial^3\lambda}{\partial y^3} + 3y^{-1}y'y'''\frac{\partial\xi}{\partial x}$$
$$+3y^{-1}y'^2y'''\frac{\partial\xi}{\partial y} + 3y^{-1}y'^2y''\frac{\partial^2\xi}{\partial x^2} + 6y^{-1}y'^2y''\frac{\partial^2\xi}{\partial x\partial y}$$
$$+3y^{-1}y'^3y''\frac{\partial^2\xi}{\partial y^2} + 3y^{-1}y'y''^2\frac{\partial\xi}{\partial y} + y^{-1}y'^2\frac{\partial^3\xi}{\partial x^3}$$
$$+3y^{-1}y'^3\frac{\partial^3\xi}{\partial x^2\partial y} + 3y^{-1}y'^2y''\frac{\partial^2\xi}{\partial x\partial y} + y^{-1}y'^2y'''\frac{\partial\xi}{\partial y}$$
$$+3y^{-1}y'^4\frac{\partial^3\xi}{\partial x\partial y^2} + 3y^{-1}y'^3y''\frac{\partial^2\xi}{\partial y^2} + y^{-1}y'^5\frac{\partial^3\xi}{\partial y^3}$$
$$+\frac{\partial^4\lambda}{\partial x^4} + 4y'\frac{\partial^4\lambda}{\partial x^3\partial y} + 6y''\frac{\partial^3\lambda}{\partial x^2\partial y} + 4y'''\frac{\partial^2\lambda}{\partial x\partial y} + y^4\frac{\partial\lambda}{\partial y}$$
$$+3y'^2\frac{\partial^4\lambda}{\partial x^2\partial y^2} + 9y'y''\frac{\partial^3\lambda}{\partial x\partial y^2} + 4y'y'''\frac{\partial^2\lambda}{\partial y^2} + 3y''^2\frac{\partial^2\lambda}{\partial y^2}$$
$$+4y'^3\frac{\partial^4\lambda}{\partial x\partial y^3} + 6y'^2y''\frac{\partial^3\lambda}{\partial y^3} + 3y'^2\frac{\partial^4\lambda}{\partial x^2\partial y^2}$$
$$+3y'y''\frac{\partial^3\lambda}{\partial x\partial y^2} + y'^4\frac{\partial^4\lambda}{\partial y^4} - 4y^{-1}y'y'''\frac{\partial\xi}{\partial x}$$
$$-4y^{-1}y'^2y'''\frac{\partial\xi}{\partial y} - 6y'''\frac{\partial^2\xi}{\partial x^2} - 12y'y'''\frac{\partial^2\xi}{\partial x\partial y}$$
$$-6y'^2y'''\frac{\partial^2\xi}{\partial y^2} - 6y''y'''\frac{\partial\xi}{\partial y} - 4y''\frac{\partial^3\xi}{\partial x^3} - 12y'y''\frac{\partial^3\xi}{\partial x^2\partial y}$$
$$-12y''^2\frac{\partial^2\xi}{\partial x\partial y} - 4y''y'''\frac{\partial\xi}{\partial y} - 12y'^2y''\frac{\partial^3\xi}{\partial x\partial y^2}$$
$$-12y'y''^2\frac{\partial^2\xi}{\partial y^2} - 4y'^3y''\frac{\partial^2\xi}{\partial y^2} - y'\frac{\partial^4\xi}{\partial x^4} - 4y'^2\frac{\partial^4\xi}{\partial x^3\partial y}$$
$$-6y'y''\frac{\partial^3\xi}{\partial x^2\partial y} - 4y'y'''\frac{\partial^2\xi}{\partial x\partial y} - y'y^4\frac{\partial\xi}{\partial y} - 3y'^3\frac{\partial^4\xi}{\partial x^2\partial y^2}$$
$$-9y'^2y''\frac{\partial^3\xi}{\partial x\partial y^2} - 4y'^2y'''\frac{\partial^2\xi}{\partial y^2} - 3y'y''^2\frac{\partial^2\xi}{\partial y^2}$$
$$-4y'^4\frac{\partial^4\xi}{\partial x\partial y^3} - 6y'^3y''\frac{\partial^3\xi}{\partial y^3} - 3y'^3\frac{\partial^4\xi}{\partial x^2\partial y^2}$$
$$-3y'^2y''\frac{\partial^3\xi}{\partial x\partial y^2} - y'^5\frac{\partial^4\xi}{\partial y^4} = 0 \qquad (9)$$

Substituting equation (1) into (9) yields the simplified form:

$$\lambda y^{-2}y'y''' - y^{-1}y'''\frac{\partial\lambda}{\partial x} - y^{-1}y'y'''\frac{\partial\lambda}{\partial y} - y^{-1}y'\frac{\partial^3\lambda}{\partial x^3}$$
$$-3y^{-1}y'^2\frac{\partial^3\lambda}{\partial x^2\partial y} - 3y^{-1}y'y''\frac{\partial^2\lambda}{\partial x\partial y} - 3y^{-1}y'^3\frac{\partial^3\lambda}{\partial x\partial y^2}$$
$$-3y^{-1}y'^2y''\frac{\partial^2\lambda}{\partial y^2} - y^{-1}y'^4\frac{\partial^3\lambda}{\partial y^3} + 3y^{-1}y'y''\frac{\partial^2\xi}{\partial x^2}$$
$$+9y^{-1}y'^2y''\frac{\partial^2\xi}{\partial x\partial y} + 6y^{-1}y'^3y''\frac{\partial^2\xi}{\partial y^2} + 3y^{-1}y'y''^2\frac{\partial\xi}{\partial y}$$
$$+y^{-1}y'^2\frac{\partial^3\xi}{\partial x^3} + 3y^{-1}y'^3\frac{\partial^3\xi}{\partial x^2\partial y} + 3y^{-1}y'^4\frac{\partial^3\xi}{\partial x\partial y^2}$$
$$+y^{-1}y'^5\frac{\partial^3\xi}{\partial y^3} + \frac{\partial^4\lambda}{\partial x^4} + 4y'\frac{\partial^4\lambda}{\partial x^3\partial y} + 6y''\frac{\partial^3\lambda}{\partial x^2\partial y}$$
$$+4y'''\frac{\partial^2\lambda}{\partial x\partial y} + 6y'^2\frac{\partial^4\lambda}{\partial x^2\partial y^2} + 12y'y''\frac{\partial^3\lambda}{\partial x\partial y^2}$$

$$+4y'y'''\frac{\partial^2\lambda}{\partial y^2}+3y''^2\frac{\partial^2\lambda}{\partial y^2}+4y'^3\frac{\partial^4\lambda}{\partial x\partial y^3}+6y'^2y''\frac{\partial^3\lambda}{\partial y^3}$$
$$+y'^4\frac{\partial^4\lambda}{\partial y^4}-6y'''\frac{\partial^2\xi}{\partial x^2}-16y'y'''\frac{\partial^2\xi}{\partial x\partial y}-10y'^2y'''\frac{\partial^2\xi}{\partial y^2}$$
$$-10y''y'''\frac{\partial\xi}{\partial y}-4y''\frac{\partial^3\xi}{\partial x^3}-18y'y''\frac{\partial^3\xi}{\partial x^2\partial y}-12y''^2\frac{\partial^2\xi}{\partial x\partial y}$$
$$-24y'^2y''\frac{\partial^3\xi}{\partial x\partial y^2}-15y'y''^2\frac{\partial^2\xi}{\partial y^2}-10y'^3y''\frac{\partial^2\xi}{\partial y^2}$$
$$-y'\frac{\partial^4\xi}{\partial x^4}-4y'^2\frac{\partial^4\xi}{\partial x^3\partial y}-4y'y'''\frac{\partial^2\xi}{\partial x\partial y}-6y'^3\frac{\partial^4\xi}{\partial x^2\partial y^2}$$
$$-4y'^4\frac{\partial^4\xi}{\partial x\partial y^3}-y'^5\frac{\partial^4\xi}{\partial y^4}=0 \tag{10}$$

From (10) we obtain the determining equations [4]:

$$-10\frac{\partial^2\xi}{\partial y^2}=0 \tag{11}$$

$$3\frac{\partial^2\lambda}{\partial y^2}-12\frac{\partial^2\xi}{\partial x\partial y}=0 \tag{12}$$

$$-y^{-1}\frac{\partial\lambda}{\partial x}+4\frac{\partial^2\lambda}{\partial x\partial y}-6\frac{\partial^2\xi}{\partial y^2}=0 \tag{13}$$

$$-3y^{-1}\frac{\partial^2\lambda}{\partial x\partial y}+3y^{-1}\frac{\partial^2\xi}{\partial y^2}+12\frac{\partial^3\lambda}{\partial x\partial y^2}-18\frac{\partial^3\xi}{\partial x^2\partial y}=0 \tag{14}$$

From the determining equations we obtain the Generator:

$$T=R_1\left(xy\frac{\partial}{\partial x}+2y^2\frac{\partial}{\partial y}\right)+R_2\left(y\frac{\partial}{\partial x}\right)+R_3\left(\frac{\partial}{\partial y}\right)$$
$$+R_4\left(y\frac{\partial}{\partial y}\right)+R_5\left(x\frac{\partial}{\partial x}\right)+R_6\left(\frac{\partial}{\partial x}\right) \tag{15}$$

From (15) we have six one parameter symmetries:

$$T_1=\frac{\partial}{\partial x} \tag{16}$$

$$T_2=\frac{\partial}{\partial y} \tag{17}$$

$$T_3=x\frac{\partial}{\partial x} \tag{18}$$

$$T_4=y\frac{\partial}{\partial x} \tag{19}$$

$$T_5=y\frac{\partial}{\partial y} \tag{20}$$

$$T_6=xy\frac{\partial}{\partial x}+2y^2\frac{\partial}{\partial y} \tag{21}$$

From the above symmetries, we consider the sub algebra [7]:

$$P_1=\frac{\partial}{\partial x} \tag{22}$$

$$P_2=x\frac{\partial}{\partial x} \tag{23}$$

$$P_3=y\frac{\partial}{\partial x} \tag{24}$$

We shall also apply the following extended prolongations up to the fourth order [6]:

$$T^{[0]}=\xi\frac{\partial}{\partial x}+\lambda\frac{\partial}{\partial y} \tag{25}$$

$$T^{[1]}=T^{[0]}+\left(\lambda'-y'\xi'\right)\frac{\partial}{\partial y'} \tag{26}$$

$$T^{[2]}=T^{[1]}+\left(\lambda''-2y''\xi'-y'\xi''\right)\frac{\partial}{\partial y''} \tag{27}$$

$$T^{[3]}=T^{[2]}+\left(\lambda'''-3y'''\xi'-3y''\xi''-y'\xi'''\right)\frac{\partial}{\partial y'''} \tag{28}$$

$$T^{[4]}=T^{[3]}+\left(\lambda^{(4)}-4y^{(4)}\xi'-6y'''\xi''-4y''\xi'''-y'\xi^{(4)}\right)\frac{\partial}{\partial y^{(4)}} \tag{29}$$

For $P_1=\frac{\partial}{\partial x}$ we have

$$P_1^{[0]}=1.\frac{\partial}{\partial x}+0.\frac{\partial}{\partial y}=\frac{\partial}{\partial x} \tag{30}$$

$$P_1^{[1]}=P_1^{[0]}+\left(0-y'.0\right)\frac{\partial}{\partial y'}=\frac{\partial}{\partial x} \tag{31}$$

$$P_1^{[2]}=P_1^{[1]}+\left(0-2y''.0-y'.0\right)\frac{\partial}{\partial y''}=\frac{\partial}{\partial x} \tag{32}$$

$$P_1^{[3]}=P_1^{[2]}+\left(0-3y'''.0-3y''.0-y'.0\right)\frac{\partial}{\partial y'''}=\frac{\partial}{\partial x} \tag{33}$$

$$P_1^{[4]}=P_1^{[3]}+\left(0-4y^{(4)}.0-6y'''.0-4y''.0-y'.0\right)\frac{\partial}{\partial y^{(4)}} \tag{34}$$

$$P_1^{[4]}=\frac{\partial}{\partial x} \tag{35}$$

From (35) we can say that

$$P_1^{[4]} = 1.\frac{\partial}{\partial x} + 0.\frac{\partial}{\partial y} \tag{36}$$

From (36) we get the characteristic:

$$\frac{dx}{1} = \frac{dy}{0}$$
$$\Rightarrow dy = 0$$
$$\Rightarrow y = l \qquad l \text{ is a constant} \tag{37}$$

For $P_2 = x\frac{\partial}{\partial x}$ we have the following prolongations

$$P_2^{[0]} = x\frac{\partial}{\partial x} + 0.\frac{\partial}{\partial y} = x\frac{\partial}{\partial x} \tag{38}$$

$$P_2^{[1]} = P_2^{[0]} + \left(0 - y'.1\right)\frac{\partial}{\partial y'} = x\frac{\partial}{\partial x} - y'\frac{\partial}{\partial y'} \tag{39}$$

$$P_2^{[2]} = P_2^{[1]} + \left(0 - 2y''.1 - 0\right)\frac{\partial}{\partial y''}$$
$$= x\frac{\partial}{\partial x} - y'\frac{\partial}{\partial y'} - 2y''\frac{\partial}{\partial y''} \tag{40}$$

$$P_2^{[3]} = P_2^{[2]} + \left(0 - 3y'''.1 - 0 - 0\right)\frac{\partial}{\partial y'''} \tag{41}$$
$$= x\frac{\partial}{\partial x} - y'\frac{\partial}{\partial y'} - 2y''\frac{\partial}{\partial y''} - 3y'''\frac{\partial}{\partial y'''}$$

$$P_2^{[4]} = P_2^{[3]} + \left(0 - 4y^{[4]}.1 - 0 - 0 - 0\right)\frac{\partial}{\partial y^{(4)}}$$
$$= x\frac{\partial}{\partial x} - y'\frac{\partial}{\partial y'} - 2y''\frac{\partial}{\partial y''} - 3y'''\frac{\partial}{\partial y'''} - 4y^{(4)}\frac{\partial}{\partial y^{(4)}} \tag{42}$$

From (42) we solve the equations

$$\frac{dx}{x} = \frac{dy'}{-y'} = \frac{dy''}{-2y''} = \frac{dy'''}{-3y'''} = \frac{dy^{(4)}}{-4y^{(4)}} \tag{43}$$

From (43) we solve the invariants as follows:

a)

$$\frac{dx}{x} = \frac{dy'}{-y}$$
$$\ln x = \ln\frac{s_1}{y'} \tag{45}$$

Or better still

$$s_1 = xy' \qquad \text{where } s_1 \text{ is a constant} \tag{46}$$

b)

$$\frac{dy'}{-y'} = \frac{dy''}{-2y''}$$
$$\ln y' = \frac{1}{2}\ln y'' + \ln t$$
$$\ln y' = \ln t\left(y''\right)^{\frac{1}{2}}$$
$$t = \frac{y'}{\left(y''\right)^{\frac{1}{2}}}$$
$$s_2 = \frac{y''}{\left(y'\right)^2} \qquad t,\ s_2 \text{ are constants.} \tag{47}$$

c)

$$\frac{dy'}{-y'} = \frac{dy'''}{-3y'''}$$
$$\ln y' = \frac{1}{3}\ln y''' + \ln t_1$$
$$\ln y' = \ln t_1\left(y'''\right)^{\frac{1}{3}}$$
$$t_1 = \frac{y'}{\left(y'''\right)^{\frac{1}{3}}}$$
$$s_3 = \frac{y'''}{\left(y'\right)^3} \qquad t_1,\ s_3 \text{ are constants} \tag{48}$$

d)

$$\frac{dy'}{-y'} = \frac{dy^{(4)}}{-4y^{(4)}}$$
$$\ln y' = \frac{1}{4}\ln y^{(4)} + \ln t_2$$
$$\ln y' = \ln t_2\left(y^{(4)}\right)^{\frac{1}{4}}$$
$$t_2 = \frac{y'}{\left(y^{(4)}\right)^{\frac{1}{4}}}$$
$$s_4 = \frac{y^{(4)}}{\left(y'\right)^4} \qquad t_2,\ s_4 \text{ are constants} \tag{49}$$

e)

$$\frac{dy''}{-2y''}=\frac{dy'''}{-3y'''}$$

$$\ln y''^{\frac{1}{2}}=\ln y'''^{\frac{1}{3}}+\ln t_3$$

$$\ln y''^{\frac{1}{2}}=\ln t_3\left(y'''\right)^{\frac{1}{3}}$$

$$t_3=\frac{y''^{\frac{1}{2}}}{\left(y'''\right)^{\frac{1}{3}}}$$

$$s_5=\frac{y'''^2}{y''^3} \qquad t_3,\ s_5 \text{ are constants} \qquad (50)$$

f)

$$\frac{dy''}{-2y''}=\frac{dy^{(4)}}{-4y^{(4)}}$$

$$\ln y''^{\frac{1}{2}}=\ln y^{(4)\frac{1}{4}}+\ln t_4$$

$$\ln y''^{\frac{1}{2}}=\ln t_4\left(y^{(4)}\right)^{\frac{1}{4}}$$

$$t_4=\frac{y''^{\frac{1}{2}}}{\left(y^{(4)}\right)^{\frac{1}{4}}}$$

$$s_6=\frac{y^{(4)}}{y''^2} \qquad t_4,\ s_6 \text{ are constants} \qquad (51)$$

g)

$$\frac{dy'''}{-3y'''}=\frac{dy^{(4)}}{-4y^{(4)}}$$

$$\ln y'''^{\frac{1}{3}}=\ln y^{(4)\frac{1}{4}}+\ln t_5$$

$$\ln y'''^{\frac{1}{3}}=\ln t_5\left(y^{(4)}\right)^{\frac{1}{4}}$$

$$t_5=\frac{y'''^{\frac{1}{3}}}{\left(y^{(4)}\right)^{\frac{1}{4}}}$$

$$s_7=\frac{y^{(4)3}}{y'''^2} \qquad t_5,\ s_7 \text{ are constants} \qquad (52)$$

For $P_3=y\frac{\partial}{\partial x}$ we have the following prolongations

$$P_3^{[0]}=y\frac{\partial}{\partial x}+0.\frac{\partial}{\partial y}=y\frac{\partial}{\partial x}$$

$$P_3^{[1]}=P_3^{[0]}+\left(0-y'\right)\frac{\partial}{\partial y'}=y\frac{\partial}{\partial x}-y'\frac{\partial}{\partial y'}$$

$$P_3^{[2]}=P_3^{[1]}+\left(0-2y''-y'.0\right)\frac{\partial}{\partial y''}=y\frac{\partial}{\partial x}-y'\frac{\partial}{\partial y'}-2y''\frac{\partial}{\partial y''}$$

$$P_3^{[3]}=P_3^{[2]}+\left(0-3y'''.1-3y''.0-y'.0\right)\frac{\partial}{\partial y'''}$$

$$=y\frac{\partial}{\partial x}-y'\frac{\partial}{\partial y'}-2y''\frac{\partial}{\partial y''}-3y'''\frac{\partial}{\partial y'''}$$

$$P_3^{[4]}=P_3^{[3]}+\left(0-4y^{(4)}.1-6y'''.0-4y''.0-y'.0\right)\frac{\partial}{\partial y^{(4)}}$$

$$P_3^{[4]}=y\frac{\partial}{\partial x}-y'\frac{\partial}{\partial y'}-2y''\frac{\partial}{\partial y''}-3y'''\frac{\partial}{\partial y'''}-4y^{(4)}\frac{\partial}{\partial y^{(4)}}$$

Clearly $P_3=y\frac{\partial}{\partial x}$ produces similar results as $P_2=x\frac{\partial}{\partial x}$.

We now apply equations (37) and (48) to reduce (1) to lower order, i.e. we use $y=l$ and $s=\frac{y'''}{\left(y'\right)^3}$ to reduce

$y^{[4]}=y^{-1}y'y'''$ to lower order as below

$$\frac{ds}{dl}=\frac{D_x\left(s\right)}{D_x\left(l\right)}=\frac{D_x\left(\frac{y'''}{\left(y'\right)^3}\right)}{D_x\left(y\right)}$$

$$\frac{ds}{dl}=\frac{\left(y'\right)^3y^{(4)}-3y'''\left(y'\right)^2y''}{\left(y'\right)^6y'}$$

$$\frac{ds}{dl}=\frac{\left(y'\right)^3y^{(4)}-3y'''\left(y'\right)^2y''}{\left(y'\right)^7}$$

$$\frac{ds}{dl}=\frac{y^{(4)}}{\left(y'\right)^4}-\frac{3y''y'''}{\left(y'\right)^5}$$

$$\frac{ds}{dl}=\frac{y^{-1}y'y'''}{\left(y'\right)^4}-\frac{3y''y'''}{\left(y'\right)^5}$$

$$\frac{ds}{dl}=\frac{y^{-1}y'y'''}{y'\left(y'\right)^3}-\frac{3y''y'''}{\left(y'\right)^5}$$

$$\frac{ds}{dl} = sl^{-1} - 3y''s(l')^{-2}$$

$$\frac{ds}{dl} = sl^{-1} - 3l''(l')^{-2}s$$

$$\frac{ds}{dl} = \left(l^{-1} - 3l''(l')^{-2}\right)s$$

$$\frac{ds}{s} = \left(l^{-1} - 3l''(l')^{-2}\right)dl \qquad (52)$$

Integrating both sides of (52) we get:

$$\int \frac{ds}{s} = \int \left(l^{-1} - 3l''(l')^{-2}\right)dl$$

$$\int \frac{ds}{s} = \int l^{-1}dl - 3\int l''(l')^{-2}\,dl$$

$$\ln s = \ln l + 3(l')^{-1}$$

$$e^{\ln s} = e^{\ln l + 3(l')^{-1}}$$

$$s = e^{\ln l}.e^{3(l')^{-1}}$$

$$s = le^{3(l')^{-1}} \qquad (53)$$

Equation (53) forms the general solution to our non-linear differential equation (1).

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