

Determining Equations of Fourth Order Nonlinear Ordinary Differential Equation

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Abstract– Determining Equations are linear partial differential equations. The equation to be solved is subjected to extension generator. The coefficient of unconstrained partial derivatives is equated to zero and since the equations are homogeneous their solutions form vector space [1]. The determining equations obtained leads to n-parameter symmetries.

Keywords– Infinitesimal Generators, Prolongation, Lie Symmetry, Ordinary Differential Equation and Determining Equation

I. WAVE EQUATION

The solution of fourth order nonlinear wave equation:

$$F(x, y, y', y'', y''', y^{(4)}) = 0 \quad (1)$$

It can be classified either as analytic or as numerical solutions using finite difference approach where the convergence of the numerical schemes depend entirely on the initial and boundary values given. We have attempted in this study to solve a special case of equation:

$$\left(yy'(y(y')^{-1})'' \right) = 0 \quad (2)$$

analytically using Lie Symmetry approach.

Equation (2) can alternatively be expressed as:

$$\left[yy' \left(\frac{y}{y'} \right)'' \right]' = 0 \quad (3)$$

The equation (3) can be decomposed in the form:

$y^{(4)} = (x, y, y', y'', y''')$ as follows:

$$4y'y^{-1}y''^2 - 4y^2y'^{-3}y''^3 + 5y^2y'^{-2}y''y''' - y'y'' - 3yy''' - y^2y'^{-1}y^{(4)} = 0 \quad (4)$$

We also notice that:

$$y^{(4)} = 4y^{-1}(y'')^2 - 4(y')^{-2}(y'')^3 + 5(y')^{-1}y''y''' - y^{-2}(y')^2y'' - 3y^{-1}y'y''' \quad (5)$$

Since the equation is fourth order differential equation, we use the fourth extension of G which from the n^{th} extension of the form [4]:

$$G^{(n)} = G \sum_{i=1}^n \left\{ \beta^{(i)} - \sum_{j=1}^i \binom{i}{j} y^{i+1-j} \alpha^{(j)} \right\} \frac{\partial}{\partial y^{(i)}} \text{ is given by:}$$

$$G^{(4)} = \alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y} + (\beta' - \alpha' y') \frac{\partial}{\partial y'} + (\beta'' - 2y'' \alpha' - y' \alpha'') \frac{\partial}{\partial y''} \quad (6)$$

$$+ (\beta''' - 3y''' \alpha' - 3y'' \alpha'' - y' \alpha''') \frac{\partial}{\partial y'''} + \beta^{(4)} - 4y^{(4)} \alpha' - 4y''' \alpha'' - y' \alpha^{(4)} \frac{\partial}{\partial y^{(4)}}$$

When $G^{(4)}$ acts on the differential equation, we have:

$$G^{(4)}[y^{(4)} - 4y^{-1}(y'')^2 + 4y^{-2}(y'')^3 - 5(y'^{-1})y''y''' + y^{-2}(y')^2y'' + 3y^{-1}y'y'''] = 0$$

On expanding, we obtain

$$\begin{aligned} & \alpha[(y^{(5)}) + 4y^{-2}(y')(y'')^2 - 8y^{-1}(y'')(y''') - 8(y^{-1})(y'')(y''') - 8(y')^{-3}(y'')^4 \\ & + 12(y')^{-2}(y'')^2(y''') + 5(y')^{-2}(y'')^2(y''') - 5(y')^{-1}(y''') + 5(y')(y'^2(y''') - 5(y')(y''')^2 \\ & - 5(y')^{-1}(y'')(y^{(4)}) - 2y^{-3}(y')^3(y'') + 2y^{-2}(y')(y'')^2 + y^{-2}(y')^2(y'' - 3y^{-2}(y')^2(y'')) \\ & + 3y^{-1}(y'')(y''') + 3y^{-1}(y')(y^{(4)})] + \beta[4y^{-2}(y'')^2 - 2y^{-3}(y')^2(y'') - 3y^{-2}(y')(y''')] \\ & + [\beta' - \alpha' y'][-8(y')^{-3}(y'')^4 + 5(y')^{-2}(y'')^2(y''') + 2y^{-2}(y')(y'')^2 + 3y^{-1}(y''')] \\ & + [\beta'' - 2y'' \alpha' - y' \alpha''][-8y^{-1}(y'')(y''') + 12(y')^{-2}(y'')^2(y''') - 5(y')^{-1}(y''') + y^{-2}(y')^2] \\ & + [\beta''' - 3y''' \alpha' - 3y'' \alpha'' - y' \alpha'''][-5(y')^{-1}(y'') + 3y^{-1}(y')] + [\beta^{(4)} - 4y^{(4)} \alpha' - 6y''' \alpha'' - 4y'' \alpha''' \\ & - y' \alpha^{(4)}][1] = 0 \end{aligned} \quad (7)$$

But we notice

$$\begin{aligned} y^{(5)} &= (y^{(4)})' \\ &= (4y^{-1}(y'')^2 - 4(y')^{-2}(y'')^3 + 5(y')^{-1}y''y''' - y^{-2}(y')^2y'' - 3y^{-1}y'y''')' \\ &= -4y^{-2}(y')(y'') + 8y^{-1}(y'')(y''') + 8(y')^{-3}(y'')^4 - 12(y')^{-2}(y'')^2(y''') \\ & - 5(y')^{-2}(y'')^2(y''') + 5(y')^{-1}(y''')^2 + 5(y')^{-1}(y'')(y^{(4)}) + 2y^{-3}(y')^3(y'') \\ & - 2y^{-2}(y')(y'')^2 - y^{-2}(y')^2(y''') + 3y^{-2}(y')^2(y''') - 3y^{-1}(y'')(y''') - 3y^{-1}(y')(y^{(4)}) \end{aligned} \quad (8)$$

Substituting (8) in (7) gives:

$$\begin{aligned}
& \alpha[(-4y^{-2}(y')(y'')^2 + 8y^{-1}(y'')(y''') + 8(y')^{-3}(y'')^4 - 12(y')^{-2}(y'')^2(y''')) \\
& - 5(y')^{-2}(y'')^2(y''') + 5(y')^{-1}(y''')^2 + 5(y')^{-1}(y'')(y^{(4)}) + 2y^{-3}(y')^3(y'') \\
& - 2y^{-2}(y')(y'')^2 - y^{-2}(y')^2(y'')^2(y''') + 3y^{-2}(y')^2(y''') - 3y^{-1}(y'')(y''') \\
& - 3y^{-1}(y')(y^{(4)}) + 4y^{-2}(y')(y'')^2 - 8y^{-1}(y'')(y''') - 8(y')^{-3}(y'')^4 + 12(y')^{-2}(y'')^2(y''') \\
& + 5(y')^{-2}(y'')^2(y''') - 5(y')^{-1}(y''')^2 - 5(y')^{-1}(y'')(y^{(4)}) - 2y^{-3}(y')^3(y'') + 2y^{-2}(y')(y'')^2 \\
& + y^{-2}(y')^2(y''') - 3y^{-2}(y')^2(y''') + 3y^{-1}(y'')(y''') + 3y^{-1}(y')(y^{(4)})] \\
& + \beta[4y^{-2}(y'')^2 - 2y^{-3}(y')^2(y'') - 3y^{-2}(y')(y''')] \\
& + [\beta' - \alpha' y'][-8(y')^{-3}(y'')^4 + 5(y')^{-2}(y'')^2(y''') + 2y^{-2}(y')(y'')^2 + 3y^{-1}(y''')] \\
& + [\beta'' - 2y''\alpha' - y'\alpha''][-8y^{-1}(y'')(y''') + 12(y')^{-2}(y'')^2(y''') - 5(y')^{-1}(y''') + y^{-2}(y')^2] \\
& + [\beta''' - 3y''' \alpha' - 3y'' \alpha'' - y' \alpha'''][-5(y')^{-1}(y'') + 3y^{-1}(y')] \\
& + [\beta^{(4)} - 4y^{(4)} \alpha' - 6y''' \alpha'' - 4y'' \alpha''' - y' \alpha^{(4)}] = 0
\end{aligned}$$

Or equivalently

$$\begin{aligned}
& -4\alpha y^{-2}(y')(y'')^2 + 8\alpha y^{-1}(y'')(y''') + 8\alpha(y')^{-3}(y'')^4 - 12\alpha(y')^{-2}(y'')^2(y''') \\
& - 5\alpha(y')^{-2}(y'')^2(y''') + 5\alpha(y')^{-1}(y''')^2 + 5\alpha(y')^{-1}(y'')(y^{(4)}) + 2\alpha y^{-3}(y')^3(y'') \\
& - 2\alpha y^{-2}(y')(y'')^2 - \alpha y^{-2}(y')^2(y'')^2(y''') + 3\alpha y^{-2}(y')^2(y''') - 3\alpha y^{-1}(y'')(y''') \\
& - 3\alpha y^{-1}(y')(y^{(4)}) + 4\alpha y^{-2}(y')(y'')^2 - 8\alpha y^{-1}(y'')(y''') - 8(y')^{-3}(y'')^4 \\
& + 12\alpha(y')^{-2}(y'')^2(y''') + 5\alpha(y')^{-2}(y'')^2(y''') - 5\alpha(y')^{-1}(y''')^2 - 5\alpha(y')^{-1}(y'')(y^{(4)}) \\
& - 2\alpha y^{-3}(y')^3(y'') + 2\alpha y^{-2}(y')(y'')^2 + \alpha y^{-2}(y')^2(y''') - 3\alpha y^{-2}(y')^2(y''') \\
& + 3\alpha y^{-1}(y'')(y''') + 3\alpha y^{-1}(y^{(4)}) + 4\beta y^{-2}(y'')^2 - 2\beta y^{-3}(y')^2(y'') - 3\beta y^{-2}(y')(y''') \\
& - 8\beta'(y')^{-3}(y'')^4 + 5\beta'(y')^{-2}(y'')^2(y''') + 2\beta' y^{-2}(y')(y'')^2 + 3\beta' y^{-1}(y''') \\
& + 8\alpha'(y')^{-2}(y'')^4 - 5\alpha'(y')^{-1}(y'')^2(y''') - 2\alpha' y^{-2}(y')^2(y'')^2 - 3\alpha' y^{-1}(y')(y''') \\
& - 8\beta'' y^{-1}(y'')(y''') + 12\beta''(y')^{-2}(y'')^2(y''') - 5\beta''(y')^{-1}(y''') + \beta'' y^{-2}(y')^2 \\
& + 16\alpha' y^{-1}(y'')^2(y''') - 24\alpha'(y')^{-2}(y'')^3(y''') + 5\alpha''(y''') - \alpha'' y^{-2}(y')^1 \\
& - 5\beta'''(y')^{-1}(y'') + 15\alpha'(y')^{-1}(y'')(y''') + 15\alpha''(y')^{-1}(y'')^2 + 5\alpha'''(y'') \\
& 3\beta''' y^{-1}(y') - 27\alpha' y^{-1}(y')(y''') - 9\alpha'' y^{-1}(y')(y'') - 3\alpha''' y^{-1}(y')^2 \\
& \beta^{(4)} - 4\alpha' y^{(4)} - 6\alpha'' y''' - 4\alpha''' y'' - \alpha^{(4)} y' = 0
\end{aligned}$$

Which can progressively be expressed as

$$\begin{aligned}
& -4\alpha y^{-2}(y')(y'')^2 + 8\alpha y^{-1}(y'')(y''') + 8\alpha(y')^{-3}(y'')^4 - 12\alpha(y')^{-2}(y'')^2(y''') \\
& -5\alpha(y')^{-2}(y'')^2(y''') + 5\alpha(y')^{-1}(y''')^2 + 2\alpha y^{-3}(y')^3(y'') \\
& -2\alpha y^{-2}(y')(y'')^2 - \alpha y^{-2}(y')^2(y'')^2(y''') + 3\alpha y^{-2}(y')^2(y''') - 3\alpha y^{-1}(y'')(y''') \\
& + 4\alpha y^{-2}(y')(y'')^2 - 8\alpha y^{-1}(y'')(y''') - 8(y')^{-3}(y'')^4 \\
& + 12\alpha(y')^{-2}(y'')^2(y''') + 5\alpha(y')^{-2}(y'')^2(y''') - 5\alpha(y')^{-1}(y''')^2 \\
& -2\alpha y^{-3}(y')^3(y'') + 2\alpha y^{-2}(y')(y'')^2 + \alpha y^{-2}(y')^2(y'')^2(y''') - 3\alpha y^{-2}(y')^2(y''') \\
& + 3\alpha y^{-1}(y'')(y''') + 4\beta y^{-2}(y'')^2 - 2\beta y^{-3}(y')^2(y'') - 3\beta y^{-2}(y')(y''') \\
& -8\beta'(y')^{-3}(y'')^4 + 5\beta'(y')^{-2}(y'')^2(y''') + 2\beta' y^{-2}(y')(y'')^2 + 3\beta' y^{-1}(y''') \\
& + 8\alpha'(y')^{-2}(y'')^4 - 5\alpha'(y')^{-1}(y'')^2(y''') - 2\alpha' y^{-2}(y')^2(y'')^2 - 3\alpha' y^{-1}(y')(y''') \\
& -8\beta'' y^{-1}(y'')(y''') + 12\beta''(y')^{-2}(y'')^2(y''') - 5\beta''(y')^{-1}(y''') + \beta'' y^{-2}(y')^2 \\
& + 16\alpha' y^{-1}(y'')^2(y''') - 24\alpha'(y')^{-2}(y'')^3(y''') + 10\alpha'(y')^{-1}(y'')(y''') - 2\alpha' y^{-2}(y')^2(y'') \\
& + 8\alpha'' y^{-1}(y')(y'')(y''') - 12\alpha'' y^{-1}(y'')^2(y''') + 5\alpha''(y''') - \alpha'' y^{-2}(y')^1 \\
& -5\beta'''(y')^{-1}(y'') + 15\alpha'(y')^{-1}(y'')(y''') + 15\alpha''(y')^{-1}(y'')^2 + 5\alpha'''(y'') \\
& + 3\beta''' y^{-1}(y') - 27\alpha' y^{-1}(y')(y''') - 9\alpha'' y^{-1}(y')(y'') - 3\alpha''' y^{-1}(y')^2 + \beta^{(4)} \\
& -16\alpha' y^{-1}(y'')^2 + 16\alpha'(y')^{-2}(y'')^3 - 20\alpha'(y')^{-1} y'' y'''' + 4\alpha' y^{-2}(y')^2 y'' + 12\alpha' y^{-1} y' y'''' \\
& -6\alpha'' y'' - 4\alpha''' y'' - \alpha^{(4)} y' = 0
\end{aligned}$$

Hence, we have

$$\begin{aligned}
& 4\beta y^{-2}(y'')^2 - 2\beta y^{-3}(y')^2(y'') - 3\beta y^{-2}(y')(y''') \\
& -8\beta'(y')^{-3}(y'')^4 + 5\beta'(y')^{-2}(y'')^2(y''') + 2\beta' y^{-2}(y')(y'')^2 + 3\beta' y^{-1}(y''') \\
& + 8\alpha'(y')^{-2}(y'')^4 - 5\alpha'(y')^{-1}(y'')^2(y''') - 2\alpha' y^{-2}(y')^2(y'')^2 - 3\alpha' y^{-1}(y')(y''') \\
& -8\beta'' y^{-1}(y'')(y''') + 12\beta''(y')^{-2}(y'')^2(y''') - 5\beta''(y')^{-1}(y''') + \beta'' y^{-2}(y')^2 \\
& + 16\alpha' y^{-1}(y'')^2(y''') - 24\alpha'(y')^{-2}(y'')^3(y''') + 10\alpha'(y')^{-1}(y'')(y''') - 2\alpha' y^{-2}(y')^2(y'') \\
& + 8\alpha'' y^{-1}(y')(y'')(y''') - 12\alpha'' y^{-1}(y'')^2(y''') + 5\alpha''(y''') - \alpha'' y^{-2}(y')^1 \\
& -5\beta'''(y')^{-1}(y'') + 15\alpha'(y')^{-1}(y'')(y''') + 15\alpha''(y')^{-1}(y'')^2 + 5\alpha'''(y'') \\
& + 3\beta''' y^{-1}(y') - 27\alpha' y^{-1}(y')(y''') - 9\alpha'' y^{-1}(y')(y'') - 3\alpha''' y^{-1}(y')^2 + \beta^{(4)} \\
& -16\alpha' y^{-1}(y'')^2 + 16\alpha'(y')^{-2}(y'')^3 - 20\alpha'(y')^{-1} y'' y'''' + 4\alpha' y^{-2}(y')^2 y'' + 12\alpha' y^{-1} y' y'''' \\
& -6\alpha'' y'' - 4\alpha''' y'' - \alpha^{(4)} y' = 0
\end{aligned}$$

(9)

We recall that primes in (9) refer to total derivatives and so the first, the second, the third and fourth total derivatives of α and β can be expressed in terms of partial derivatives as follows:

$$\alpha' = \frac{\partial \alpha}{\partial x} + y' \frac{\partial \alpha}{\partial y}$$

$$\alpha'' = \frac{\partial^2 \alpha}{\partial x^2} + 2y' \frac{\partial^2 \alpha}{\partial x \partial y} + y'^2 \frac{\partial^2 \alpha}{\partial y^2} + y'' \frac{\partial \alpha}{\partial y}$$

$$\alpha''' = \frac{\partial^3 \alpha}{\partial x^3} + 3y' \frac{\partial^3 \alpha}{\partial x^2 \partial y} + 3y'' \frac{\partial^2 \alpha}{\partial x \partial y} + y''' \frac{\partial \alpha}{\partial y} + 3y'^2 \frac{\partial^3 \alpha}{\partial x \partial y^2} + 3y' y'' \frac{\partial^2 \alpha}{\partial y^2} + y'^3 \frac{\partial^3 \alpha}{\partial y^3}$$

$$\begin{aligned} \alpha^4 = & \frac{\partial^4 \alpha}{\partial x^4} + 4y' \frac{\partial^4 \alpha}{\partial x^3 \partial y} + 6y'' \frac{\partial^3 \alpha}{\partial x^2 \partial y} + 4y''' \frac{\partial^2 \alpha}{\partial x \partial y} + y^{(4)} \frac{\partial \alpha}{\partial y} \\ & + 3y'^2 \frac{\partial^4 \alpha}{\partial x^2 \partial y^2} + 9y' y'' \frac{\partial^3 \alpha}{\partial x \partial y^2} + 4y' y''' \frac{\partial^2 \alpha}{\partial y^2} + 3y'^2 \frac{\partial^2 \alpha}{\partial y^2} + 4y'^3 \frac{\partial^4 \alpha}{\partial x \partial y^3} \\ & + 6y'^2 y'' \frac{\partial^3 \alpha}{\partial y^3} + 3y'^2 \frac{\partial^4 \alpha}{\partial x^2 \partial y^2} + 3y' y'' \frac{\partial^3 \alpha}{\partial x \partial y^2} + y'^4 \frac{\partial^4 \alpha}{\partial y^4} \end{aligned}$$

And so is true for β .

Therefore, we express equation (9) as

$$\begin{aligned}
& 4\beta y^{-2}(y'')^2 - 2\beta y^{-3}(y')^2(y'') - 3\beta y^{-2}(y')(y''') - 8\left[\frac{\partial\beta}{\partial x} + y'\frac{\partial\beta}{\partial y}\right](y')^{-3}(y'')^4 + \\
& 5\left[\frac{\partial\beta}{\partial x} + y'\frac{\partial\beta}{\partial y}\right](y')^{-2}(y'')^2(y''') + 2\left[\frac{\partial\beta}{\partial x} + y'\frac{\partial\beta}{\partial y}\right]y^{-2}(y')(y'')^2 + 3\left[\frac{\partial\beta}{\partial x} + y'\frac{\partial\beta}{\partial y}\right]y^{-1}(y''') \\
& + 8\left[\frac{\partial\alpha}{\partial x} + y'\frac{\partial\alpha}{\partial y}\right](y')^{-2}(y'')^4 - 5\left[\frac{\partial\alpha}{\partial x} + y'\frac{\partial\alpha}{\partial y}\right](y')^{-1}(y'')^2(y''') - 2\left[\frac{\partial\alpha}{\partial x} + y'\frac{\partial\alpha}{\partial y}\right]y^{-2}(y')^2(y'')^2 \\
& - 3\left[\frac{\partial\alpha}{\partial x} + y'\frac{\partial\alpha}{\partial y}\right]y^{-1}(y')(y''') - 8\left[\frac{\partial^2\beta}{\partial x^2} + 2y'\frac{\partial^2\beta}{\partial x\partial y} + y'^2\frac{\partial^2\beta}{\partial y^2} + y''\frac{\partial\beta}{\partial y}\right]y^{-1}(y'')(y''') \\
& + 12\left[\frac{\partial^2\beta}{\partial x^2} + 2y'\frac{\partial^2\beta}{\partial x\partial y} + y'^2\frac{\partial^2\beta}{\partial y^2} + y''\frac{\partial\beta}{\partial y}\right](y')^{-2}(y'')^2(y''') \\
& - 5\left[\frac{\partial^2\beta}{\partial x^2} + 2y'\frac{\partial^2\beta}{\partial x\partial y} + y'^2\frac{\partial^2\beta}{\partial y^2} + y''\frac{\partial\beta}{\partial y}\right](y')^{-1}(y''') \\
& + \left[\frac{\partial^2\beta}{\partial x^2} + 2y'\frac{\partial^2\beta}{\partial x\partial y} + y'^2\frac{\partial^2\beta}{\partial y^2} + y''\frac{\partial\beta}{\partial y}\right]y^{-2}(y')^2 + 16\left[\frac{\partial\alpha}{\partial x} + y'\frac{\partial\alpha}{\partial y}\right]y^{-1}(y'')^2(y''') \\
& - 24\left[\frac{\partial\alpha}{\partial x} + y'\frac{\partial\alpha}{\partial y}\right](y')^{-2}(y'')^3(y''') + 10\left[\frac{\partial\alpha}{\partial x} + y'\frac{\partial\alpha}{\partial y}\right](y')^{-1}(y'')(y''') - 2\left[\frac{\partial\alpha}{\partial x} + y'\frac{\partial\alpha}{\partial y}\right]y^{-2}(y')^2(y'') \\
& + 8\left[\frac{\partial^2\alpha}{\partial x^2} + 2y'\frac{\partial^2\alpha}{\partial x\partial y} + y'^2\frac{\partial^2\alpha}{\partial y^2} + y''\frac{\partial\alpha}{\partial y}\right]y^{-1}(y')(y'')(y''') - 12\left[\frac{\partial^2\alpha}{\partial x^2} + 2y'\frac{\partial^2\alpha}{\partial x\partial y} + y'^2\frac{\partial^2\alpha}{\partial y^2} + y''\frac{\partial\alpha}{\partial y}\right] \\
& y^{-1}(y'')^2(y''') + 5\left[\frac{\partial^2\alpha}{\partial x^2} + 2y'\frac{\partial^2\alpha}{\partial x\partial y} + y'^2\frac{\partial^2\alpha}{\partial y^2} + y''\frac{\partial\alpha}{\partial y}\right](y''') \\
& - \left[\frac{\partial^2\alpha}{\partial x^2} + 2y'\frac{\partial^2\alpha}{\partial x\partial y} + y'^2\frac{\partial^2\alpha}{\partial y^2} + y''\frac{\partial\alpha}{\partial y}\right]y^{-2}(y')^1 \\
& - 5\left[\frac{\partial^3\beta}{\partial x^3} + 3y'\frac{\partial^3\beta}{\partial x^2\partial y} + 3y''\frac{\partial^2\beta}{\partial x\partial y} + y'''\frac{\partial\beta}{\partial y} + 3y'^2\frac{\partial^3\beta}{\partial x\partial y^2} + 3y'y''\frac{\partial^2\beta}{\partial y^2} + y'^3\frac{\partial^3\beta}{\partial y^3}\right](y')^{-1}(y'') \\
& + 15\left[\frac{\partial\alpha}{\partial x} + y'\frac{\partial\alpha}{\partial y}\right](y')^{-1}(y'')(y''') + 15\left[\frac{\partial^2\alpha}{\partial x^2} + 2y'\frac{\partial^2\alpha}{\partial x\partial y} + y'^2\frac{\partial^2\alpha}{\partial y^2} + y''\frac{\partial\alpha}{\partial y}\right](y')^{-1}(y'')^2
\end{aligned}$$

$$\begin{aligned}
& + 5 \left[\frac{\partial^3 \alpha}{\partial x^3} + 3y' \frac{\partial^3 \alpha}{\partial x^2 \partial y} + 3y'' \frac{\partial^2 \alpha}{\partial x \partial y} + y'''' \frac{\partial \alpha}{\partial y} + 3y'^2 \frac{\partial^3 \alpha}{\partial x \partial y^2} + 3y' y'' \frac{\partial^2 \alpha}{\partial y^2} + y'^3 \frac{\partial^3 \alpha}{\partial y^3} \right] (y''') \\
& + 3 \left[\frac{\partial^3 \beta}{\partial x^3} + 3y' \frac{\partial^3 \beta}{\partial x^2 \partial y} + 3y'' \frac{\partial^2 \beta}{\partial x \partial y} + y'''' \frac{\partial \beta}{\partial y} + 3y'^2 \frac{\partial^3 \beta}{\partial x \partial y^2} + 3y' y'' \frac{\partial^2 \beta}{\partial y^2} + y'^3 \frac{\partial^3 \beta}{\partial y^3} \right] y^{-1} (y') \\
& - 27 \left[\frac{\partial \alpha}{\partial x} + y' \frac{\partial \alpha}{\partial y} \right] y^{-1} (y') (y''') - 9 \left[\frac{\partial^2 \alpha}{\partial x^2} + 2y' \frac{\partial^2 \alpha}{\partial x \partial y} + y'^2 \frac{\partial^2 \alpha}{\partial y^2} + y'' \frac{\partial \alpha}{\partial y} \right] y^{-1} (y') (y'')^2 \\
& - 3 \left[\frac{\partial^3 \alpha}{\partial x^3} + 3y' \frac{\partial^3 \alpha}{\partial x^2 \partial y} + y'''' \frac{\partial \alpha}{\partial y} + 3y'^2 \frac{\partial^3 \alpha}{\partial x \partial y^2} + 3y' y'' \frac{\partial^2 \alpha}{\partial y^2} + y'^3 \frac{\partial^3 \alpha}{\partial y^3} \right] y^{-1} (y')^2 \\
& + \frac{\partial^4 \beta}{\partial x^4} + 4y' \frac{\partial^4 \beta}{\partial x^3 \partial y} + 6y'' \frac{\partial^3 \beta}{\partial x^2 \partial y} + 4y'''' \frac{\partial^2 \beta}{\partial x \partial y} + y^{(4)} \frac{\partial \beta}{\partial y} + 3y'^2 \frac{\partial^4 \beta}{\partial x^2 \partial y^2} + 9y' y'' \frac{\partial^3 \beta}{\partial x \partial y^2} \\
& + 4y' y'''' \frac{\partial^2 \beta}{\partial y^2} + 3y'^2 \frac{\partial^2 \beta}{\partial y^2} + 4y'^3 \frac{\partial^4 \beta}{\partial x \partial y^3} + 6y'^2 y'' \frac{\partial^3 \beta}{\partial y^3} + 3y'^2 \frac{\partial^4 \beta}{\partial x^2 \partial y^2} \\
& - 16 \left[\frac{\partial \alpha}{\partial x} + y' \frac{\partial \alpha}{\partial y} \right] y^{-1} (y'')^2 + 16 \left[\frac{\partial \alpha}{\partial x} + y' \frac{\partial \alpha}{\partial y} \right] (y')^{-2} (y'')^3 - 20 \left[\frac{\partial \alpha}{\partial x} + y' \frac{\partial \alpha}{\partial y} \right] (y')^{-1} y'' y'''' \\
& + 4 \left[\frac{\partial \alpha}{\partial x} + y' \frac{\partial \alpha}{\partial y} \right] y^{-2} (y')^2 y'' + 12 \left[\frac{\partial \alpha}{\partial x} + y' \frac{\partial \alpha}{\partial y} \right] y^{-1} y' y'''' - 6 \left[\frac{\partial^2 \alpha}{\partial x^2} + 2y' \frac{\partial^2 \alpha}{\partial x \partial y} + y'^2 \frac{\partial^2 \alpha}{\partial y^2} + y'' \frac{\partial \alpha}{\partial y} \right] y'''' \\
& - 4 \left[\frac{\partial^3 \alpha}{\partial x^3} + 3y' \frac{\partial^3 \alpha}{\partial x^2 \partial y} + 3y'' \frac{\partial^2 \alpha}{\partial x \partial y} + y'''' \frac{\partial \alpha}{\partial y} + 3y'^2 \frac{\partial^3 \alpha}{\partial x \partial y^2} + 3y' y'' \frac{\partial^2 \alpha}{\partial y^2} + y'^3 \frac{\partial^3 \alpha}{\partial y^3} \right] y'''' \\
& - \left[\frac{\partial^4 \alpha}{\partial x^4} + 4y' \frac{\partial^4 \alpha}{\partial x^3 \partial y} + 6y'' \frac{\partial^3 \alpha}{\partial x^2 \partial y} + 4y'''' \frac{\partial^2 \alpha}{\partial x \partial y} + y^{(4)} \frac{\partial \alpha}{\partial y} \right. \\
& + 3y'^2 \frac{\partial^4 \alpha}{\partial x^2 \partial y^2} + 9y' y'' \frac{\partial^3 \alpha}{\partial x \partial y^2} + 4y' y'''' \frac{\partial^2 \alpha}{\partial y^2} + 3y'^2 \frac{\partial^2 \alpha}{\partial y^2} + 4y'^3 \frac{\partial^4 \alpha}{\partial x \partial y^3} \\
& \left. + 6y'^2 y'' \frac{\partial^3 \alpha}{\partial y^3} + 3y'^2 \frac{\partial^4 \alpha}{\partial x^2 \partial y^2} + 3y' y'' \frac{\partial^3 \alpha}{\partial x \partial y^2} + y'^4 \frac{\partial^4 \alpha}{\partial y^4} \right] [y'] = 0
\end{aligned} \tag{10}$$

Expanding the equation (10) we have

$$\begin{aligned}
& 4\beta y^{-2} (y'')^2 - 2\beta y^{-3} (y')^2 (y'') - 3\beta y^{-2} (y') (y''') - 8(y')^{-3} (y'')^4 \frac{\partial \beta}{\partial x} \\
& - 8(y')^{-2} (y'')^4 \frac{\partial \beta}{\partial y} + 5(y')^{-2} (y'')^2 (y''') \frac{\partial \beta}{\partial x} + (y')^{-1} (y'')^2 (y''') \frac{\partial \beta}{\partial y} \\
& + 2y^{-2} (y') (y'')^2 \frac{\partial \beta}{\partial x} + 2y^{-2} (y')^2 (y'')^2 \frac{\partial \beta}{\partial y} + 3y^{-1} (y''') \frac{\partial \beta}{\partial x} + 3y^{-1} (y') (y''') \frac{\partial \beta}{\partial y}
\end{aligned}$$

$$\begin{aligned}
& + 8(y')^{-2}(y'')^4 \frac{\partial \alpha}{\partial x} + 8(y')^{-1}(y'')^4 \frac{\partial \alpha}{\partial y} - 5(y')(y''')^2(y''''') \frac{\partial \alpha}{\partial x} - 5(y'')^2(y''''') \frac{\partial \alpha}{\partial y} \\
& - 2y^{-2}(y')^2(y''')^2 \frac{\partial \alpha}{\partial x} - 2y^{-2}(y')^3(y'')^2 \frac{\partial \alpha}{\partial y} - 3y^{-1}(y')(y''') \frac{\partial \alpha}{\partial x} - 3y^{-1}(y')^2(y''') \frac{\partial \alpha}{\partial y} \\
& - 8y^{-1}(y'')(y''''') \frac{\partial^2 \beta}{\partial x^2} - 16y^{-1}(y')(y'')(y''''') \frac{\partial^2 \beta}{\partial x \partial y} - 8y^{-1}(y')^2(y'')(y''''') \frac{\partial^2 \beta}{\partial y^2} \\
& - 8y^{-1}(y'')^2(y''''') \frac{\partial \beta}{\partial y} + 12(y')^{-2}(y'')^3(y''''') \frac{\partial \beta}{\partial y} - 5(y')^{-1}(y''''') \frac{\partial^2 \beta}{\partial x^2} \\
& - 10(y''''') \frac{\partial^2 \beta}{\partial x \partial y} - 5(y')^1(y''''') \frac{\partial^2 \beta}{\partial y^2} - 5(y')^{-1}(y'')(y''''') \frac{\partial \beta}{\partial y} + y^{-2}(y')^2 \frac{\partial^2 \beta}{\partial x^2} \\
& + 2y^{-2}(y')^3 \frac{\partial^2 \beta}{\partial x \partial y} + y^{-2}(y')^4 \frac{\partial^2 \beta}{\partial y^2} + y^{-2}(y')^2(y'') \frac{\partial \beta}{\partial y} + 16y^{-1}(y'')^2(y''''') \frac{\partial \alpha}{\partial x} \\
& + 16y^{-1}(y')(y'')^2(y''''') \frac{\partial \alpha}{\partial y} - 24(y')^{-2}(y''')^3(y''''') \frac{\partial \alpha}{\partial x} - 24(y')^{-1}(y''')^3(y''''') \frac{\partial \alpha}{\partial y} \\
& + 10(y')^{-1}(y'')(y''''') \frac{\partial \alpha}{\partial x} + 10(y'')(y''''') \frac{\partial \alpha}{\partial y} - 2y^{-2}(y')^2(y'') \frac{\partial \alpha}{\partial x} - 2y^{-1}(y')^3(y'') \frac{\partial \alpha}{\partial y} \\
& + 8y^{-1}(y')(y'')(y''''') \frac{\partial^2 \alpha}{\partial x^2} + 16y^{-1}(y')^2(y'')(y''''') \frac{\partial^2 \alpha}{\partial x \partial y} + 8y^{-1}(y')^3(y'')(y''''') \frac{\partial^2 \alpha}{\partial y^2} \\
& + 8y^{-1}(y')(y'')^2(y''''') \frac{\partial \alpha}{\partial y} - 12(y')^{-1}(y'')^2(y''''') \frac{\partial^2 \alpha}{\partial x^2} - 24(y'')^2(y''''') \frac{\partial^2 \alpha}{\partial x \partial y} \\
& - 12(y')^1(y'')^2(y''''') \frac{\partial^2 \alpha}{\partial y^2} - 12(y')^{-1}(y''')^3(y''''') \frac{\partial \alpha}{\partial y} + 5(y''''') \frac{\partial^2 \alpha}{\partial x^2} + 10(y')(y''''') \frac{\partial^2 \alpha}{\partial x \partial y} \\
& + 5(y')^2(y''''') \frac{\partial^2 \alpha}{\partial y^2} + 5(y'')(y''''') \frac{\partial \alpha}{\partial y} - y^{-2}(y')^1 \frac{\partial^2 \alpha}{\partial x^2} - 2y^{-2}(y')^2 \frac{\partial^2 \alpha}{\partial x \partial y} \\
& - y^{-2}(y')^3 \frac{\partial^2 \alpha}{\partial y^2} - y^{-2}(y')^1(y'') \frac{\partial \alpha}{\partial y} - 5(y')^{-1}(y'') \frac{\partial^3 \beta}{\partial x^3} - 15(y'') \frac{\partial^3 \beta}{\partial x^2 \partial y} \\
& - 5(y')^{-1}(y'')^2 \frac{\partial^2 \beta}{\partial x \partial y} + 5(y')^{-1}(y'')(y''''') \frac{\partial \beta}{\partial y} - 15(y')(y'') \frac{\partial^3 \beta}{\partial x \partial y^2} - 15(y'')^2 \frac{\partial^2 \beta}{\partial y^2} \\
& - 5(y')^2(y'') \frac{\partial^3 \beta}{\partial y^3} + 15(y')^{-1}(y'')(y''''') \frac{\partial \alpha}{\partial x} + 15(y'')(y''''') \frac{\partial \alpha}{\partial y} + 15(y')(y'')^2 \frac{\partial^2 \alpha}{\partial x^2} \\
& + 30(y'')^2 \frac{\partial^2 \alpha}{\partial x \partial y} + 15(y')^1(y'')^2 \frac{\partial^2 \alpha}{\partial y^2} + 15(y')^{-1}(y'')^3 \frac{\partial \alpha}{\partial y} + 5(y'') \frac{\partial^3 \alpha}{\partial x^3} + 15(y')(y'') \frac{\partial^3 \alpha}{\partial x^2 \partial y} \\
& + 15(y'')^2 \frac{\partial^2 \alpha}{\partial x \partial y} + 5(y'')(y''''') \frac{\partial \alpha}{\partial y} + 15(y')^2(y'') \frac{\partial^3 \alpha}{\partial x \partial y^2} + 15(y')(y'')^2 \frac{\partial^2 \alpha}{\partial y^2} + 15(y')^3(y'') \frac{\partial^3 \alpha}{\partial y^3} \\
& + 3y^{-1}(y') \frac{\partial^3 \beta}{\partial x^3} + 9y^{-1}(y')^2 \frac{\partial^3 \beta}{\partial x^2 \partial y} + 9y^{-1}(y')(y'') \frac{\partial^2 \beta}{\partial x \partial y} + 3y^{-1}(y')(y''''') \frac{\partial \beta}{\partial y}
\end{aligned}$$

$$\begin{aligned}
& + 9y^{-1}(y')^3 \frac{\partial^3 \beta}{\partial x \partial y^2} + 9y^{-1}(y')^2(y'') \frac{\partial^2 \beta}{\partial y^2} + 3y^{-1}(y')^4 \frac{\partial^3 \beta}{\partial y^3} - 27y^{-1}(y')(y''') \frac{\partial \alpha}{\partial x} \\
& - 27y^{-1}(y')^2(y''') \frac{\partial \alpha}{\partial y} - 9y^{-1}(y')(y'') \frac{\partial^2 \alpha}{\partial x^2} - 18y^{-1}(y')^2(y'') \frac{\partial^2 \alpha}{\partial x \partial y} - 9y^{-1}(y')^3(y'') \frac{\partial^2 \alpha}{\partial y^2} \\
& - 9y^{-1}(y')(y'')^2 \frac{\partial \alpha}{\partial y} - 3y^{-1}(y')^2 \frac{\partial^3 \alpha}{\partial x^3} - 9y^{-1}(y')^3 \frac{\partial^3 \alpha}{\partial x^2 \partial y} - 9y^{-1}(y')^2(y'') \frac{\partial^2 \alpha}{\partial x \partial y} \\
& - 3y^{-1}(y')^2(y''') \frac{\partial \alpha}{\partial y} - 9y^{-1}(y')^4 \frac{\partial^3 \alpha}{\partial x \partial y^2} + 9y^{-1}(y')^3(y'') \frac{\partial^2 \alpha}{\partial y^2} - 3y^{-1}(y')^5 \frac{\partial^3 \alpha}{\partial y^3} \\
& \frac{\partial^4 \beta}{\partial x^4} + 4y' \frac{\partial^4 \beta}{\partial x^3 \partial y} + 6y'' \frac{\partial^3 \beta}{\partial x^2 \partial y} + 4y''' \frac{\partial^2 \beta}{\partial x \partial y} + 4y^{-1}(y'')^2 \frac{\partial \beta}{\partial y} - 4(y')(y'')^3 \frac{\partial \beta}{\partial y} + 5(y')^{-1}(y'')(y''') \frac{\partial \beta}{\partial y} \\
& - y^{-2}(y')^2(y'') \frac{\partial \beta}{\partial y} - 3y^{-1}(y')(y''') \frac{\partial \beta}{\partial y} + 3(y')^2 \frac{\partial^4 \beta}{\partial x^2 \partial y^2} + 9(y')(y'') \frac{\partial^3 \beta}{\partial x \partial y^2} + 4(y')(y''') \frac{\partial^2 \beta}{\partial y^2} \\
& + 3(y'')^2 \frac{\partial^2 \beta}{\partial y^2} + 4(y')^3 \frac{\partial^4 \beta}{\partial x \partial y^3} + 6(y')^2 y'' \frac{\partial^3 \beta}{\partial y^3} + 3(y')^2 \frac{\partial^4 \beta}{\partial x^2 \partial y^2} + 3(y')(y'') \frac{\partial^3 \beta}{\partial x \partial y^2} + 4(y')^3 \frac{\partial^4 \beta}{\partial x \partial y} \\
& + 6(y')^2 y'' \frac{\partial^3 \beta}{\partial y^3} + 3(y')^2 \frac{\partial^4 \beta}{\partial x^2 \partial y^2} + 3(y')(y'') \frac{\partial^3 \beta}{\partial x \partial y^2} + (y')^4 \frac{\partial^4 \beta}{\partial y^4} - 16y^{-1}(y'')^2 \frac{\partial \alpha}{\partial x} - 16y^{-1}(y')(y'')^2 \frac{\partial \alpha}{\partial y} \\
& + 16(y')^{-2}(y''')^3 \frac{\partial \alpha}{\partial x} + 16(y')^{-1}(y''')^3 \frac{\partial \alpha}{\partial y} - 20(y')^{-1}(y'')(y''') \frac{\partial \alpha}{\partial x} + 20(y'')(y''') \frac{\partial \alpha}{\partial y} \\
& + 4y^{-2}(y')^2(y'') \frac{\partial \alpha}{\partial x} + 4y^{-2}(y')^3(y'') \frac{\partial \alpha}{\partial y} + 12y^{-1}(y')(y''') \frac{\partial \alpha}{\partial x} + 12y^{-1}(y')^2(y''') \frac{\partial \alpha}{\partial y} \\
& - 6(y''') \frac{\partial^2 \alpha}{\partial x^2} - 12(y')(y''') \frac{\partial^2 \alpha}{\partial x \partial y} - 6(y')^2(y''') \frac{\partial^2 \alpha}{\partial y^2} - 6(y'')(y''') \frac{\partial \alpha}{\partial y} - 4(y'') \frac{\partial^3 \alpha}{\partial x^3} - 12(y')(y'') \frac{\partial^3 \alpha}{\partial x^2 \partial y} \\
& - 12(y'')^2 \frac{\partial^2 \alpha}{\partial x \partial y} - 4(y'')(y''') \frac{\partial \alpha}{\partial y} - 12(y')^2(y'') \frac{\partial^3 \alpha}{\partial x \partial y^2} - 12(y')(y'')^2 \frac{\partial^2 \alpha}{\partial y^2} - 4(y')^3(y'') \frac{\partial^3 \alpha}{\partial y^3} - (y') \frac{\partial^4 \alpha}{\partial x^4} \\
& - 4(y')^2 \frac{\partial^4 \alpha}{\partial x^3 \partial y} - 6(y')(y'') \frac{\partial^3 \alpha}{\partial x^2 \partial y} - 4(y')(y''') \frac{\partial^2 \alpha}{\partial x \partial y} - 4y^{-1}(y')(y'')^2 \frac{\partial \beta}{\partial y} + 4(y')^{-1}(y'')^3 \frac{\partial \beta}{\partial y} \\
& - 5(y'')(y''') \frac{\partial \beta}{\partial y} + y^{-2}(y')^3 \frac{\partial \beta}{\partial y} + 3y^{-1}(y')^2(y''') \frac{\partial \beta}{\partial y} - 3(y')^3 \frac{\partial^4 \alpha}{\partial x^2 \partial y^2} - 9(y')^2(y'') \frac{\partial^3 \alpha}{\partial x \partial y^2} \\
& - 4(y')^2(y''') \frac{\partial^2 \alpha}{\partial y^2} - 3(y')(y'')^2 \frac{\partial^2 \alpha}{\partial y^2} - 4(y')^4 \frac{\partial^4 \alpha}{\partial x \partial y^3} - 6(y')^3(y'') \frac{\partial^3 \alpha}{\partial y^3} - 3(y')^3 \frac{\partial^4 \alpha}{\partial x^2 \partial y^2} - 3(y')^2(y'') \frac{\partial^3 \alpha}{\partial x \partial y^2} \\
& - (y')^5 \frac{\partial^4 \alpha}{\partial y^4} = 0
\end{aligned}$$

(11)

The equation is an identity in x, y, y', y'', y, y''' that is it holds for any arbitrary choices of x, y, y', y'', y, y''' [3]. Since α and β are functions of x and y only, we must equate the coefficients of the powers of y', y'', y, y''' and their combinations to zero. We obtain the following systems of partial differential equations known as Determining Equations [3], [4]:

$$(y')^3 y'' y''': 8y^{-1} \frac{\partial^2 \alpha}{\partial y^2} = 0 \quad (12)$$

$$(y')^2 y'' y''': -8y^{-1} \frac{\partial^2 \beta}{\partial y^2} + 16y^{-1} \frac{\partial^2 \alpha}{\partial x \partial y} = 0 \quad (13)$$

$$(y')^1 y'' y''': -16y^{-1} \frac{\partial^2 \beta}{\partial x \partial y} + 8y^{-1} \frac{\partial^2 \alpha}{\partial x^2} = 0 \quad (14)$$

$$(y')^0 y'' y''': -8y^{-1} \frac{\partial^2 \beta}{\partial x^2} + 10 \frac{\partial \alpha}{\partial y} + 5 \frac{\partial \alpha}{\partial y} - 5 \frac{\partial \alpha}{\partial y} + 5 \frac{\partial \alpha}{\partial y} - 20 \frac{\partial \alpha}{\partial y} - 6 \frac{\partial \alpha}{\partial y} - 4 \frac{\partial \alpha}{\partial y} = 0$$

$$\Rightarrow -8y^{-1} \frac{\partial^2 \beta}{\partial x^2} + 5 \frac{\partial \alpha}{\partial y} - 5 \frac{\partial \beta}{\partial y} = 0 \quad (15)$$

II. CONCLUSION

In this paper, we have subjected the Nonlinear Wave Equation to extended generator and constructed the Determining Equations.

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