A Study on the Joint Maximal Numerical Range of Aluthge Transform

O. S. Cyprian
Department of Mathematics. Kabarak University
P.O. Box Private Bag - 20157
Kabarak, Kenya
cypriansakwa@gmail.com

***ABSTRACT*—The Aluthge transform $\widetilde{T}$ of a bounded linear operator T on a complex Hilbert space X is the operator $|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. Here, $T = U|T|$ is any polar decomposition of T with U a partial isometry and $|T| = (T^*T)^{\frac{1}{2}}$. This study of the Aluthge transform $\widetilde{T}$ was introduced and studied by Aluthge in his study of p-hyponormal operators. Since its conception, this notion has received much attention for a single operator T. In order to understand the joint behaviour of Aluthge transform of several operarors $T_1, ..., T_m$, researchers such as Cyprian have studied the Aluthge transform of an $m-$tuple operator $T = (T_1, ..., T_m)$. For instance, the properties of the joint essential numerical range of Aluthge transform for an $m-$tuple operator $T = (T_1, ..., T_m)$ were studied by Cyprian. However, nothing is known about the joint maximal numerical range of Aluthge transform $\widetilde{T}$ of an $m-$tuple operator $T = (T_1, ..., T_m)$. This paper focuses on the study of the properties of the joint maximal numerical range of Aluthge transform for an $m-$tuple operator $T = (T_1, ..., T_m)$. This study will help in the development of the research on hyponormal operators and semi-hyponormal operators.**

***Index Terms*—Aluthge transform $\widetilde{T}$, Hilbert space, Joint Maximal numerical range $\widetilde{T}$, Maximal Numerical range of $\widetilde{T}$.**

1 INTRODUCTION

The joint maximal numerical range of Aluthge transform Max$W_m(\widetilde{T})$ of several operators $T = (T_1, ..., T_m)$ is defined as the set of all complex numbers r_k for which there exist a sequence $\{x_n\}$ of unit vectors in X such that $\langle \widetilde{T}_k x_n, x_n\rangle \to r_k$ and $\|\widetilde{T}_k x_n\| \to \|\widetilde{T}_k\|$ for $0 \le k \le m$.
In the case $k = 1$, it becomes the maximal numerical range of Aluthge transform $\widetilde{T}$, Max$W(\widetilde{T})$, of a single operator T.

This paper establishes some of the properties of the set Max$W_m(\widetilde{T})$. This study is an extension of the study of the joint numerical range of Aluthge transform $W_m(\widetilde{T})$ studied by Cyprian, Aywa and Chikamai [3] among other researchers. Throughout this paper, $B(X)$ denotes the algebra of all bounded linear operators acting on a complex Hilbert space X. Recall that the Aluthge transform $\widetilde{T}$ of T is the operator $T = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. See Aluthge[1] for this and more. Note here that $T = U|T|$ is any polar decomposition of T with U a partial isometry and $|T| = (T^*T)^{\frac{1}{2}}$. Here, a linear operator $T^* \in B(X)$ denotes the adjoint of an operator $T \in B(X)$ and is defined by the relation $\langle Tx, y\rangle = \langle x, T^*y\rangle \;\forall\, y, x \in X$.

The following section is brief survey of the theory of the maximal numerical range of Aluthge transform.

2 MAXIMAL NUMERICAL RANGE OF ALUTHGE TRANSFORM

Stampfli [5] introduced and studied the concept of maximal numerical range of a bounded operator T on $B(X)$ and used it to derive an identity for the norm of a derivation. Recall here that a derivation on a Hilbert space X is a linear transformation $\delta : X \to X$ that satisfies $\delta(xy) = x\delta(y) + \delta(x)y \;\forall\, x, y \in X$. A derivation δ is said to be an inner derivation if for a fixed x we have $\delta : y \to xy - yx$. For an operator $T \in B(X)$, the inner derivation is denoted and defined as $\delta_T(Y) = TY - YT$ where $Y \in B(X)$. Stampfli [5] determined the norm of an inner derivation and showed that $\|\delta_T\| = 2\inf\{\|T - \lambda\| \;: \lambda \in \mathbb{C}\}$.

The maximal numerical range of an operator T is denoted by Max$W(T)$ and defined as
Max$W(T) = \{r \in \mathbb{C} : \langle Tx_n, x_n\rangle \to r$, where $x_n \in X$; $\|x_n\| = 1$ and $\|Tx_n\| \to \|T\|\}$. Here, the operator norm $\|T\|$ is defined as $\|T\| = \sup\{\|Tx\| : \|x\| = 1\}$. Stampfli [5] proved that $\|\delta_T\| = 2\|T\|$ if and only if $0 \in$Max$W(T)$.

Several other properties of the set Max$W(T)$ are known. For instance, it is clear from the following theorem that the set Max$W(T)$ is nonempty, closed and convex.

Theorem 1. *The set Max$W(T)$ is nonempty, closed and convex subset of $\overline{W(T)}$.*

See Stampli [5] for the proof.

Unlike the numerical range $W(T)$, the maximal numerical range, Max$W(T)$ does not satisfy the power inequality. That is,
$|$Max$W(T^n)| \not\le |$Max$W(T)|^n$ for $n = 1, 2, ...$ Also, unlike the set $W(T)$, the set Max$W(T)$ is unstable under translation. See Stampfli [5]

In 2007, Guoxing, Liu and Li [4] studied the essential numerical range and maximal numerical range of the Aluthge transform and proved several interesting results on maximal numerical range of Aluthge transform Max$W(\widetilde{T})$. Among other results, Guoxing, Liu and Li obtained relationships between Max$W(T)$ and Max$W(\widetilde{T})$ as shown in the following theorem.

Theorem 2. *Suppose $T \in B(X)$.*
1) *Max$W(T) \subset \overline{W(\widetilde{T})}$.*
2) *If $\|T\| = \|\widetilde{T}\|$, then Max$W(\widetilde{T}) \subset$ Max$W(T)$*

See Guoxing, Liu and Li [4] for the proof.

Guoxing, Liu and Li also proved the following theorem.

Theorem 3. *Max$W(\widetilde{T} - \alpha) =$ Max$W(\widetilde{T}^{(*)} - \alpha)$ for all $\alpha \in \mathbb{C}$ and $T \in B(X)$.*

See Guoxing, Liu and Li [4] for the proof.

3 Joint Maximal Numerical Range of Aluthge Transform

The properties of the joint essential numerical range of Aluthge transform for an $m-$tuple operator $T = (T_1, ..., T_m)$ were studied by Cyprian in [2]. This was as a result of the need to understand the joint behaviour of Aluthge transform of several operarors $T_1, ..., T_m$. This section, being and extension of this research of Aluthge transform, establishes some of the properties of the set Max$W_m(\widetilde{T})$. We begin with the following theorem.

Theorem 4. *Let $T = (T_1, ..., T_m) \in B(X)$ and $\widetilde{T} = |T|^{\frac{1}{2}} \cup |T|^{\frac{1}{2}}$. Max$W_m(T) \subset \overline{W_m(\widetilde{T})}$.*

Proof.. Assume that $\|T\| = 1$ and let $r = (r_1, ..., r_m) \in$ Max$W_m(T)$. There exists a sequence $\{x_n\} \in X$ of unit vectors such that

$$\lim_{n\to\infty} \|Tx_n\| = 1 \text{ and } \lim_{n\to\infty} \langle Tx_n, x_n\rangle = r$$

for $T = (T_1, ..., T_m) \in B(X)$. This implies that

$$\lim_{n\to\infty} \||T|^{1/2}x_n\| = 1 \text{ and } \lim_{n\to\infty} \|(1 - |T|)x_n\| = 0.$$

Therefore,

$$\begin{aligned}
&\lim_{n\to\infty} |\langle Tx_n, x_n\rangle - \langle \widetilde{T}|T|^{1/2}x_n, |T|^{1/2}x_n\rangle| \\
&= \lim_{n\to\infty} |\langle Tx_n, x_n\rangle - \langle U|T|x_n, |T|x_n\rangle| \\
&= \lim_{n\to\infty} |\langle Tx_n, (1 - |T|)x_n\rangle| \\
&\leq \lim_{n\to\infty} \|Tx_n\|\|(1 - |T|)x_n\| = 0.
\end{aligned}$$

Thus, $\lim_{n\to\infty} \langle \widetilde{T}|T|^{1/2}x_n, |T|^{1/2}x_n\rangle| = r.$

Letting $z_n = |T|^{1/2}x_n/\||T|^{1/2}x_n\|$ then $\{z_n\}$ is a sequence of unit vectors and

$$\lim_{n\to\infty} \langle \widetilde{T}z_n, z_n\rangle = r.$$

Thus $r = (r_1, ..., r_m) \in \overline{W_m(\widetilde{T})}$. □

Theorem 5. *Let $T = (T_1, ..., T_m) \in B(X)$ and $\widetilde{T} = |T|^{\frac{1}{2}} \cup |T|^{\frac{1}{2}}$. If $0 \in$ Max$W_m(\widetilde{T})$ then $\|\widetilde{T}\|^2 + |r|^2 \leq \|\widetilde{T} + r\|^2$ for any $r = (r_1, ..., r_m) \in \mathbb{C}^m$.*

Proof.. Let $0 \in$ Max$W_m(\widetilde{T})$. Then there exists a sequence $\{x_n\} \in X$ such that $\langle \widetilde{T}_k x_n, x_n\rangle \to 0$, $\|x_n\| = 1$ and $\|\widetilde{T}_k x_n\| \to \|\widetilde{T}_k\|$ for $0 \leq k \leq m$. Note that

$$\begin{aligned}
\|\widetilde{T}_k\|^2 + |r_k|^2 &= \lim_{n\to\infty} \|(\widetilde{T}_k + r_k)x_n\|^2 \\
&\leq \|\widetilde{T}_k + r_k\|^2,\ 0 \leq k \leq m.
\end{aligned}$$

Now,

$$\sum_{k=1}^{n} \|\widetilde{T}_k\|^2 + \sum_{k=1}^{n} |r_k|^2 \leq \sum_{k=1}^{n} \|\widetilde{T}_k + r_k\|^2.$$

Thus $\|\widetilde{T}_k\|^2 + |r_k|^2 \leq \|\widetilde{T}_k + r_k\|^2$ for any $r = (r_1, ..., r_m) \in \mathbb{C}^m$. □

We now show the relation between the essential norm $\|\widetilde{T}_k\|_e$ and $\|\widetilde{T}_k\|$.

Proposition 1. *If $\|\widetilde{T}_k\| > \|\widetilde{T}_k\|_e$, then*

Max$W_m(\widetilde{T}) = \{\langle \widetilde{T}_k x_n, x_n\rangle : \|x_n\| = 1$ *and* $\|\widetilde{T}_k x_n\| = \|\widetilde{T}_k\ \|\}$.

Proof.. Since $\|\widetilde{T}_k^*\widetilde{T}_k\| > \|\widetilde{T}_k\|_e^2 = \|\widetilde{T}_k^*\widetilde{T}_k\|_e$, there exists a finite rank projection P commuting with $\widetilde{T}_k^*\widetilde{T}_k$ such that $\|\widetilde{T}_k^*\widetilde{T}_k(I - P)\| < \|\widetilde{T}_k^*\widetilde{T}_k\|$, where I denotes the identity operator. Then $\|\widetilde{T}_k(I - P)\| < \|\widetilde{T}_k\|$. □

We require the following result to complete this proof.

Lemma 1. *Let P be a compact finite rank projection commuting with $\widetilde{T}_k^*\widetilde{T}_k$ such that $\|\widetilde{T}_k^*\widetilde{T}_k(I - P)\| < \|\widetilde{T}_k^*\widetilde{T}_k\|$. Then Max$W_m(\widetilde{T}) =$ Max$W_m(\widetilde{T}P)$.*

Proof.. We first show that Max$W_m(\widetilde{T}P) \subseteq$ Max$W_m(\widetilde{T})$. Let $P \in B(X)$ be an infinite dimensional projection such that $P\widetilde{T}_k^*\widetilde{T}_k P \in \mathcal{K}(X)$. There is thus an orthonormal sequence $\{x_n\} \in X$ such that $Px_n = x_n\ \forall\ n$.
Let $K = (K_1, ..., K_m) \in \mathcal{K}(X)$. For any $K_j : j \in [1, m]$, $P\widetilde{T}_k^*\widetilde{T}_k P = K_j + r_k P$ and thus $\langle (P\widetilde{T}_k^*\widetilde{T}_k P - r_k P)x_n, x_n\rangle = \langle K_j x_n, x_n\rangle$ implying $\langle \widetilde{T}_k^*\widetilde{T}_k x_n, x_n\rangle = r_k + \langle K_j x_n, x_n\rangle$. From the orthonormality of sequence $\{x_n\}$, we get $K_j x_n$ converging weakly to 0 in norm as $n \to \infty$, $j \in [1, m]$. Therefore, $\langle \widetilde{T}_k^*\widetilde{T}_k x_n, x_n\rangle \longrightarrow r_k$ as $n \to \infty$ implying $r_k \in$ Max$W_m(\widetilde{T})$.
To complete the proof, it is sufficient to prove that Max$W_m(\widetilde{T}) \subseteq$ Max$W_m(\widetilde{T}P)$. Let $r_k \in$ Max$W_m(\widetilde{T})$. This implies that there is a sequence $\{x_n\}$ of unit vectors such that $\|\widetilde{T}_k x_n\| \to \|\widetilde{T}_k\|$ and $\langle \widetilde{T}_k x_n, x_n\rangle \to r_k$; $1 \leq k \leq m\}$. Notice that $\|\widetilde{T}_k P\| = \|\widetilde{T}_k\|$. Using

$$\begin{aligned}
\|\widetilde{T}_k\|^2 \geq \|\widetilde{T}_k^*\widetilde{T}_k x_n\| &\geq \langle \widetilde{T}_k^*\widetilde{T}_k x_n, x_n\rangle \\
&= \|\widetilde{T}_k x_n\|^2 \to \|\widetilde{T}_k\|^2
\end{aligned}$$

we obtain $\|\widetilde{T}_k^*\widetilde{T}_k x_n \to \|\widetilde{T}_k\|^2$. Let $x_n = \alpha_n y_n + \beta_n z_n$ with $\|y_n\| = 1 = \|z_n\|, |\alpha_n|^2 + |\beta_n|^2 = 1, Py_n = y_n$ and $Pz_n = 0$. Since $\widetilde{T}_k^*\widetilde{T}_k$ commutes with P we have,

$$\begin{aligned}
\|\widetilde{T}_k\|^2 &\geq |\alpha_n|^2\|(\widetilde{T}_k^*\widetilde{T}_k)^{1/2}y_n\|^2 + |\beta_n|^2\|(\widetilde{T}_k^*\widetilde{T}_k)^{1/2}z_n\|^2 \\
&= \|(\widetilde{T}_k^*\widetilde{T}_k)^{1/2}(\alpha_n y_n) + (\widetilde{T}_k^*\widetilde{T}_k)^{1/2}(\beta_n z_n)\|^2 \\
&= \|(\widetilde{T}_k^*\widetilde{T}_k)^{1/2}x_n\|^2 = \|\widetilde{T}_k x_n\|^2 \to \|\widetilde{T}_k\|^2.
\end{aligned}$$

And

$$\begin{aligned}
\|(\widetilde{T}_k^*\widetilde{T}_k)^{1/2}z_n\|^2 &= \|\widetilde{T}_k z_n\|^2 = \|\widetilde{T}_k(I-P)z_n\|^2 \\
&\leq \|\widetilde{T}_k(I-P)\|^2 \\
&< \|\widetilde{T}_k\|^2 \text{ implying that } \lim \beta_n = 0.
\end{aligned}$$

Therefore, $\|\widetilde{T}_k y_n\| \to \|\widetilde{T}_k\|$ and $\langle \widetilde{T}_k y_n, y_n\rangle \to r_k$. Thus $r_k \in \mathrm{Max}W_m(\widetilde{T}P)$. □

ACKNOWLEDGMENT

I thank all friends, family members and colleagues for valuable comments and all support related to the material in this paper.

REFERENCES

[1] A. Aluthge, On p-hyponormal operators for $0 < p < 1$. *Integral Equations and Operator Theory*, **13** (1990) , 307-315.

[2] O. S. Cyprian, A note on the joint essential numerical range of Aluthge transform, *Int. J. of Pure and Applied Math* **118(3)** (2018), 573-579, **doi:** 10.12732/ijpam.v118i3.7

[3] O. S. Cyprian, S. Aywa, L. Chikamai, Some properties of the joint numerical range of the Aluthge transform, *Int. J. of Pure and Applied Math* **118(2)** (2018), 165-172, **doi:** 10.12732/ijpam.v118i2.3

[4] G. Ji, N. Liu, Z.Li, Essential numerical range and maximal numerical range of the Aluthge transform, *Linear and Multilinear Algebra*, **55** (2007), 315-322.

[5] J. G. Stampfli, The norm of derivation, *Pac. J. Math.*, **33** (1970), 737-747.