Determining Equations for the Third Order Non-linear Differential Equation: $y''' = yy'' + y'^2$

Rodgers K. Adenyah [1] **, Elizabeth Kiburu** [2]
Department of Mathematics and Computer Science, Pwani University, P.O. Box 195, 80108 Kilifi, Kenya

***Abstract*— Determining equations of a differential equation are partial differential equations that are used to obtain the generator of infinitesimal transformation for a non-linear differential equation. In this paper we obtain the determining equations for the third order non-linear differential equation $y''' = yy'' + y'^2$. The determining equations play a big role in obtaining infinitesimal symmetries, which are later used to reduce high ordered non-linear differential equations to first order.**

***Keywords*– Determining Equations, Lie Theory, Differential Equation, Generator and Infinitesimal Transformation**

I. INTRODUCTION

Differential equations are used to model numerous phenomenon in our world, from the spread of infectious diseases to the behavior of tidal waves [2], [3]. Naturally, the study of differential equations plays a vital role in the physical sciences. Most of these equations are non-linear [5]. When a non-linear ordinary differential equation admits a given number of symmetries then it becomes possible to transform it to a linear system [1]. This is when Determining equations become key. Determining equations are used in Lie Theory [4] which is an extremely powerful algorithm method for solving differential equations especially non-linear ordinary differential equations.

II. OBTAINING THE DETERMINING EQUATIONS FOR THE NON-LINEAR DIFFERENTIAL EQUATION $y''' = yy'' + y'^2$

Let us consider the third order nonlinear differential equation:

$$y''' = yy'' + y'^2 \tag{1}$$

We seek to obtain the determining equations for the above nonlinear differential equation by using the third extension [6], [7] below:

$$E^{[3]} = \rho\frac{\partial}{\partial x} + \omega\frac{\partial}{\partial y} + (\omega' - \rho' y')\frac{\partial}{\partial y'} + (\omega'' - 2y''\rho' - y'\rho'')\frac{\partial}{\partial y''}$$
$$+ (\omega''' - 3y'''\rho' - 3y''\rho'' - y'\rho''')\frac{\partial}{\partial y'''}. \tag{2}$$

We let $E^{[3]}$ above act on our nonlinear differential equation (1) so as to have:

$$E^{[3]}\left[y''' - yy'' - y'^2\right] = 0$$

Which yields

$$\Rightarrow$$
$$\left[\rho\frac{\partial}{\partial x} + \omega\frac{\partial}{\partial y} + (\omega' - \rho' y')\frac{\partial}{\partial y'} + (\omega'' - 2y''\rho' - y'\rho'')\frac{\partial}{\partial y''}\right.$$
$$\left. + (\omega''' - 3y'''\rho' - 3y''\rho'' - y'\rho''')\frac{\partial}{\partial y'''}\right](y''' - yy'' - y'^2) = 0 \tag{3}$$

Expanding equation (3) yields:

$$\rho\frac{\partial}{\partial x}(y''' - yy'' - y'^2) + \omega\frac{\partial}{\partial y}(y''' - yy'' - y'^2)$$
$$+ (\omega' - \rho' y')\frac{\partial}{\partial y'}(y''' - yy'' - y'^2)$$
$$+ (\omega'' - 2y''\rho' - y'\rho'')\frac{\partial}{\partial y''}(y''' - yy'' - y'^2)$$
$$+ (\omega''' - 3y'''\rho' - 3y''\rho'' - y'\rho''')\frac{\partial}{\partial y'''}(y''' - yy'' - y'^2) = 0 \tag{4}$$

Which further yields

$$\rho\left[y'''' -\left(yy'''+y'y''\right)-2y'y''\right]+\omega\left[0-\left(y''(1)-0\right)\right]$$
$$+\left(\omega'-\rho'y'\right)\left[0-0-2y'\right]$$
$$+\left(\omega''-2y''\rho'-y'\rho''\right)\left[-y\right]$$
$$+\left(\omega'''-3y'''\rho'-3y''\rho''-y'\rho'''\right)\left[1\right]=0 \qquad (5)$$

Equation (5) simplifies to

$$\rho\left[\frac{d}{dx}\left(y'''\right)-yy'''-y'y''-2y'y''\right]$$
$$+\omega\left[-y''\right]+\left(\omega'-\rho'y'\right)\left[-2y'\right]$$
$$+\left(\omega''-2y''\rho'-y'\rho''\right)\left[-y\right]$$
$$+\omega'''-3y'''\rho'-3y''\rho''-y'\rho'''=0 \qquad (6)$$

Equation (6) yields

$$\rho\left[\frac{d}{dx}\left(yy''+y'^2\right)-yy'''-y'y''-2y'y''\right]$$
$$-y''\omega-2y'\omega'+2y'^2\rho'-y\omega''+2yy''\rho'+yy'\rho''$$
$$+\omega'''-3y'''\rho'-3y''\rho''-y'\rho'''=0 \qquad (7)$$

Equation (7) subsequently yields the below equations:

$$\rho\left[yy'''+y'y''+2y'y''-yy'''-y'y''-2y'y''\right]$$
$$-y''\omega-2y'\omega'+2y'^2\rho'-y\omega''+2yy''\rho'+yy'\rho''$$
$$+\omega'''-3y'''\rho'-3y''\rho''-y'\rho'''=0$$
$$\Rightarrow$$
$$\rho\left[y\left(yy''+y'^2\right)+y'y''+2y'y''-y\left(yy''+y'^2\right)-y'y''-2y'y''\right]$$
$$-y''\omega-2y'\omega'+2y'^2\rho'-y\omega''+2yy''\rho'+yy'\rho''$$
$$+\omega'''-3\left(yy''+y'^2\right)\rho'-3y''\rho''-y'\rho'''=0$$

$$\Rightarrow$$
$$\rho\left[y^2y''+yy'^2+y'y''+2y'y''-y^2y''-yy'^2-y'y''-2y'y''\right]$$
$$-y''\omega-2y'\omega'+2y'^2\rho'-y\omega''+2yy''\rho'+yy'\rho''+\omega'''$$
$$-3yy''\rho'-3y'^2\rho'-3y''\rho''-y'\rho'''=0$$

$$\Rightarrow$$
$$\rho y^2y''+\rho yy'^2+\rho y'y''+2\rho y'y''-\rho y^2y''-\rho yy'^2-\rho y'y''$$
$$-2\rho y'y''-y''\omega-2y'\omega'+2y'^2\rho'-y\omega''+2\rho'yy''+yy'\rho''$$
$$+\omega'''-3\rho'yy''-3\rho'y'^2-3\rho''y''-\rho'''y'=0$$

Which simplifies to

$$-y''\omega-2y'\omega'-y'^2\rho'-y\omega''-\rho'yy''+yy'\rho''$$
$$+\omega'''-3\rho''y''-y'\rho'''=0 \qquad (8)$$

Expressing the derivatives of ω and ρ in terms of their partial derivatives we have

$$\omega'=\frac{\partial\omega}{\partial x}+y'\frac{\partial\omega}{\partial y}\left(from\ d\left(x\right)=\left(\frac{\partial\omega}{\partial x}\right)dx+\left(\frac{\partial\omega}{\partial y}\right)dy\right) \qquad (9)$$

$$\omega''=\frac{\partial}{\partial x}\left(\frac{\partial\omega}{\partial x}+y'\frac{\partial\omega}{\partial y}\right)+\frac{\partial}{\partial y}\left(\frac{\partial\omega}{\partial x}+y'\frac{\partial\omega}{\partial y}\right)y'$$

$$\omega''=\frac{\partial^2\omega}{\partial x^2}+y'\frac{\partial^2\omega}{\partial x\partial y}+y''\frac{\partial\omega}{\partial y}+y'\frac{\partial^2\omega}{\partial x\partial y}+y'^2\frac{\partial^2\omega}{\partial y^2}+0$$

$$\omega''=\frac{\partial^2\omega}{\partial x^2}+2y'\frac{\partial^2\omega}{\partial x\partial y}+y'^2\frac{\partial^2\omega}{\partial y^2}++y''\frac{\partial\omega}{\partial y} \qquad (10)$$

$$\omega'''=\frac{\partial}{\partial x}\left(\frac{\partial^2\omega}{\partial x^2}+2y'\frac{\partial^2\omega}{\partial x\partial y}+y''\frac{\partial\omega}{\partial y}+y'^2\frac{\partial^2\omega}{\partial y^2}\right)$$
$$+y'\frac{\partial}{\partial y}\left(\frac{\partial^2\omega}{\partial x^2}+2y'\frac{\partial^2\omega}{\partial x\partial y}+y''\frac{\partial\omega}{\partial y}+y'^2\frac{\partial^2\omega}{\partial y^2}\right)$$

$$\omega'''=\frac{\partial^3\omega}{\partial x^3}+2y'\frac{\partial^2\omega}{\partial x\partial x\partial y}+2y''\frac{\partial^2\omega}{\partial x\partial y}+y''\frac{\partial^2\omega}{\partial x\partial y}$$
$$+y'''\frac{\partial\omega}{\partial y}+y'^2\frac{\partial^3\omega}{\partial x\partial y^2}+2y'y''\frac{\partial^2\omega}{\partial y^2}+y'\frac{\partial^3\omega}{\partial y\partial x^2}$$
$$+2y'^2\frac{\partial^3\omega}{\partial y\partial x\partial y}+0+y'^3\frac{\partial^3\omega}{\partial y^3}+0$$

$$\omega'''=\frac{\partial^3\omega}{\partial x^3}+3y'\frac{\partial^3\omega}{\partial x^2\partial y}+3y''\frac{\partial^2\omega}{\partial x\partial y}+y'''\frac{\partial\omega}{\partial y}$$
$$+3y'^2\frac{\partial^3\omega}{\partial x\partial y^2}+3y'y''\frac{\partial^2\omega}{\partial y^2}+y'^3\frac{\partial^3\omega}{\partial y^3} \qquad (11)$$

Similarly, for ρ we have

$$\rho' = \frac{\partial\rho}{\partial x} + y'\frac{\partial\rho}{\partial y}\left(from\ d(x) = \left(\frac{\partial\rho}{\partial x}\right)dx + \left(\frac{\partial\rho}{\partial y}\right)dy\right) \tag{12}$$

$$\rho'' = \frac{\partial}{\partial x}\left(\frac{\partial\rho}{\partial x} + y'\frac{\partial\rho}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\rho}{\partial x} + y'\frac{\partial\rho}{\partial y}\right)y'$$

$$\rho'' = \frac{\partial^2\rho}{\partial x^2} + y'\frac{\partial^2\rho}{\partial x\partial y} + y''\frac{\partial\rho}{\partial y} + y'\frac{\partial^2\rho}{\partial x\partial y} + y'^2\frac{\partial^2\rho}{\partial y^2} + 0$$

$$\rho'' = \frac{\partial^2\rho}{\partial x^2} + 2y'\frac{\partial^2\rho}{\partial x\partial y} + y'^2\frac{\partial^2\rho}{\partial y^2} + +y''\frac{\partial\rho}{\partial y} \tag{13}$$

$$\rho''' = \frac{\partial}{\partial x}\left(\frac{\partial^2\rho}{\partial x^2} + 2y'\frac{\partial^2\rho}{\partial x\partial y} + y''\frac{\partial\rho}{\partial y} + y'^2\frac{\partial^2\rho}{\partial y^2}\right)$$

$$+y'\frac{\partial}{\partial y}\left(\frac{\partial^2\rho}{\partial x^2} + 2y'\frac{\partial^2\rho}{\partial x\partial y} + y''\frac{\partial\rho}{\partial y} + y'^2\frac{\partial^2\rho}{\partial y^2}\right)$$

$$\rho''' = \frac{\partial^3\rho}{\partial x^3} + 2y'\frac{\partial^2\rho}{\partial x\partial x\partial y} + 2y''\frac{\partial^2\rho}{\partial x\partial y} + y''\frac{\partial^2\rho}{\partial x\partial y}$$

$$+y'''\frac{\partial\rho}{\partial y} + y'^2\frac{\partial^3\rho}{\partial x\partial y^2} + 2y'y''\frac{\partial^2\rho}{\partial y^2} + y'\frac{\partial^3\rho}{\partial y\partial x^2}$$

$$+2y'^2\frac{\partial^3\rho}{\partial y\partial x\partial y} + 0 + y'^3\frac{\partial^3\rho}{\partial y^3} + 0$$

$$\rho''' = \frac{\partial^3\rho}{\partial x^3} + 3y'\frac{\partial^3\rho}{\partial x^2\partial y} + 3y''\frac{\partial^2\rho}{\partial x\partial y} + y'''\frac{\partial\rho}{\partial y}$$

$$+3y'^2\frac{\partial^3\rho}{\partial x\partial y^2} + 3y'y''\frac{\partial^2\rho}{\partial y^2} + y'^3\frac{\partial^3\rho}{\partial y^3} \tag{14}$$

Subjecting equation (8) to the above partial derivatives we obtain:

$$-\omega y'' - 2y'\left(\frac{\partial\omega}{\partial x} + y'\frac{\partial\omega}{\partial y}\right) - y'^2\left(\frac{\partial\rho}{\partial x} + y'\frac{\partial\rho}{\partial y}\right)$$

$$-y\left(\frac{\partial^2\omega}{\partial x^2} + 2y'\frac{\partial^2\omega}{\partial x\partial y} + y'^2\frac{\partial^2\omega}{\partial y^2} + +y''\frac{\partial\omega}{\partial y}\right)$$

$$-yy''\left(\frac{\partial\rho}{\partial x} + y'\frac{\partial\rho}{\partial y}\right) + yy'\left(\frac{\partial^2\rho}{\partial x^2} + 2y'\frac{\partial^2\rho}{\partial x\partial y} + y'^2\frac{\partial^2\rho}{\partial y^2} + +y''\frac{\partial\rho}{\partial y}\right)$$

$$+\frac{\partial^3\omega}{\partial x^3} + 3y'\frac{\partial^3\omega}{\partial x^2\partial y} + 3y''\frac{\partial^2\omega}{\partial x\partial y} + y'''\frac{\partial\omega}{\partial y} + 3y'^2\frac{\partial^3\omega}{\partial x\partial y^2} + 3y'y''\frac{\partial^2\omega}{\partial y^2} + y'^3\frac{\partial^3\omega}{\partial y^3}$$

$$-3\left(\frac{\partial^2\rho}{\partial x^2} + 2y'\frac{\partial^2\rho}{\partial x\partial y} + y'^2\frac{\partial^2\rho}{\partial y^2} + +y''\frac{\partial\rho}{\partial y}\right)y''$$

$$-y'(\frac{\partial^3\rho}{\partial x^3} + 3y'\frac{\partial^3\rho}{\partial x^2\partial y} + 3y''\frac{\partial^2\rho}{\partial x\partial y} + y'''\frac{\partial\rho}{\partial y}$$

$$+3y'^2\frac{\partial^3\rho}{\partial x\partial y^2} + 3y'y''\frac{\partial^2\rho}{\partial y^2} + y'^3\frac{\partial^3\rho}{\partial y^3}) = 0 \tag{15}$$

Equation (15) simplifies to

$$-\omega y'' - 2y'\frac{\partial\omega}{\partial x} - 2y'^2\frac{\partial\omega}{\partial y} - y'^2\frac{\partial\rho}{\partial x} - y'^3\frac{\partial\rho}{\partial y}$$

$$-y\frac{\partial^2\omega}{\partial x^2} - 2yy'\frac{\partial^2\omega}{\partial x\partial y} - yy'^2\frac{\partial^2\omega}{\partial y^2} - yy''\frac{\partial\omega}{\partial y}$$

$$-yy''\frac{\partial\rho}{\partial x} - yy'y''\frac{\partial\rho}{\partial y} + yy'\frac{\partial^2\rho}{\partial x^2} + 2yy'^2\frac{\partial^2\rho}{\partial x\partial y} + yy'^3\frac{\partial^2\rho}{\partial y^2}$$

$$+yy'y''\frac{\partial\rho}{\partial y} + \frac{\partial^3\omega}{\partial x^3} + 3y'\frac{\partial^3\omega}{\partial x^2\partial y} + 3y''\frac{\partial^2\omega}{\partial x\partial y} + y'''\frac{\partial\omega}{\partial y}$$

$$+3y'^2\frac{\partial^3\omega}{\partial x\partial y^2} + 3y'y''\frac{\partial^2\omega}{\partial y^2} + y'^3\frac{\partial^3\omega}{\partial y^3} - 3y''\frac{\partial^2\rho}{\partial x^2}$$

$$-6y'y''\frac{\partial^2\rho}{\partial x\partial y} - 3y'^2y''\frac{\partial^2\rho}{\partial y^2} - 3y''^2\frac{\partial\rho}{\partial y} - y'\frac{\partial^3\rho}{\partial x^3}$$

$$-3y'^2\frac{\partial^3\rho}{\partial x^2\partial y} - 3y'y''\frac{\partial^2\rho}{\partial x\partial y} - y'y'''\frac{\partial\rho}{\partial y} - 3y'^3\frac{\partial^3\rho}{\partial x\partial y^2}$$

$$-3y'^2y''\frac{\partial^2\rho}{\partial y^2} - y'^4\frac{\partial^3\rho}{\partial y^3} = 0 \tag{16}$$

Equation (16) simplifies to

$$-\omega y'' - 2y'\frac{\partial \omega}{\partial x} - 2y'^2\frac{\partial \omega}{\partial y} - y'^2\frac{\partial \rho}{\partial x} - y'^3\frac{\partial \rho}{\partial y}$$

$$-y\frac{\partial^2 \omega}{\partial x^2} - 2yy'\frac{\partial^2 \omega}{\partial x \partial y} - yy'^2\frac{\partial^2 \omega}{\partial y^2} - yy''\frac{\partial \omega}{\partial y}$$

$$-yy''\frac{\partial \rho}{\partial x} - yy'y''\frac{\partial \rho}{\partial y} + yy'\frac{\partial^2 \rho}{\partial x^2} + 2yy'^2\frac{\partial^2 \rho}{\partial x \partial y} + yy'^3\frac{\partial^2 \rho}{\partial y^2}$$

$$+yy'y''\frac{\partial \rho}{\partial y} + \frac{\partial^3 \omega}{\partial x^3} + 3y'\frac{\partial^3 \omega}{\partial x^2 \partial y} + 3y''\frac{\partial^2 \omega}{\partial x \partial y} + y'''\frac{\partial \omega}{\partial y}$$

$$+3y'^2\frac{\partial^3 \omega}{\partial x \partial y^2} + 3y'y''\frac{\partial^2 \omega}{\partial y^2} + y'^3\frac{\partial^3 \omega}{\partial y^3} - 3y''\frac{\partial^2 \rho}{\partial x^2}$$

$$-6y'y''\frac{\partial^2 \rho}{\partial x \partial y} - 3y'^2 y''\frac{\partial^2 \rho}{\partial y^2} - 3y''^2\frac{\partial \rho}{\partial y} - y'\frac{\partial^3 \rho}{\partial x^3}$$

$$-3y'^2\frac{\partial^3 \rho}{\partial x^2 \partial y} - 3y'y''\frac{\partial^2 \rho}{\partial x \partial y} - y'y'''\frac{\partial \rho}{\partial y} - 3y'^3\frac{\partial^3 \rho}{\partial x \partial y^2}$$

$$-3y'^2 y''\frac{\partial^2 \rho}{\partial y^2} - y'^4\frac{\partial^3 \rho}{\partial y^3} = 0 \qquad (17)$$

From (17) we obtain the following determining equations

$$y'' \ : \ -\omega - 3\frac{\partial^2 \rho}{\partial x^2} + 3\frac{\partial^2 \omega}{\partial x \partial y}$$

$$y' \ : \ -2\frac{\partial \omega}{\partial x} + 3\frac{\partial^3 \omega}{\partial x^2 \partial y} - \frac{\partial^3 \rho}{\partial x^3}$$

$$y'^2 \ : \ -2\frac{\partial \omega}{\partial y} - \frac{\partial \rho}{\partial x} + 3\frac{\partial^3 \omega}{\partial x \partial y^2} - 3\frac{\partial^3 \rho}{\partial x^2 \partial y}$$

$$y'^3 \ : \ -\frac{\partial \rho}{\partial y} + \frac{\partial^3 \omega}{\partial y^3} - 3\frac{\partial^3 \rho}{\partial x \partial y^2}$$

$$y \ : \ -\frac{\partial^2 \omega}{\partial x^2}$$

$$yy' \ : \ -2\frac{\partial^2 \omega}{\partial x \partial y} + \frac{\partial^2 \rho}{\partial x^2}$$

$$yy'^2 \ : \ -\frac{\partial^2 \omega}{\partial y^2} + 2\frac{\partial^2 \rho}{\partial x \partial y}$$

$$yy'' \ : \ -\frac{\partial \omega}{\partial y} - \frac{\partial \rho}{\partial x}$$

$$yy'y'' : \ -\frac{\partial \rho}{\partial y} + \frac{\partial \rho}{\partial y}$$

$$yy'^3 \ : \ -\frac{\partial^2 \rho}{\partial y^2}$$

$$y''' \ : \ \frac{\partial \omega}{\partial y}$$

$$y'y'' \ : \ 3\frac{\partial^2 \omega}{\partial y^2} - 9\frac{\partial^2 \rho}{\partial x \partial y}$$

$$y'^2 y'' \ : \ -3\frac{\partial^2 \rho}{\partial y^2} - 3\frac{\partial^2 \rho}{\partial y^2} = -6\frac{\partial^2 \rho}{\partial y^2}$$

$$y''^2 \ : \ -3\frac{\partial \rho}{\partial y}$$

$$y'y''' \ : \ -\frac{\partial \rho}{\partial y}$$

$$y'^4 \ : \ \frac{\partial^3 \rho}{\partial y^3}$$

III. CONCLUSION

In this paper, we use a generator of order n to obtain extension of Third order [7], and further use the extension of third order to obtain the determining equations for a third order nonlinear ordinary differential equation. The determining equations can be used to obtain symmetries for the nonlinear differential equation. The symmetries are used to obtain an analytic solution to our non-linear differential equation. The determination of analytical solutions for ordinary differential equations is the utmost importance in many fields of applied sciences.

REFERENCES

[1] Baikov, V.A. et al (1989). *Approximate symmetries of equations with a small parameter. Math*. US.S.R-Sb 64.

[2] Bluman, G.W., Anco, S.C. (2002). *Symmetry and Integration Methods for Differential Equations;* Springer: New York, NY, USA.

[3] Bluman, G.W., Kumei, S. (1989). *Symmetries and differential Equations;* Springer- Verlag: New York, NY, USA.

[4] Dresner, L. (1999). *Applications of Lie's Theory of Ordinary and Partial Differential Equations;* London, Institute of Physics.

[5] Ibragimov N.H (1999). *Elementary Lie Group Analysis and Ordinary Differential Equations*., John Wiley and sons Ltd, England.

[6] Mahomed, F.M, et al. (1990). *Symmetry Lie Algebras of nth order ordinary differential equations*.J. Math. Anal. Appl. 151.80-107

[7] Olver, P.J. (1993). *Applications of Lie Groups to Differential Equations*, (Newyork, Springer-Verlag).