Chatter Stability Boundary Frequencies of a Three Tooth End Milling Process

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***Abstract*– Stability chart of a three tooth milling process was generated using the Fargue type approximation method and numerically shown to permit only two types of loss of stability which are the secondary Hopf and flip bifurcations. This result is in conformity with theoretical result that has also been validated experimentally thus validating the generated chart. Another contribution of this work is that stability the condition of periodic retarded dynamical systems is established through the use of similarity transformation of second rank tensors. The expressions 'stability boundary frequencies' and 'chatter frequencies' are used synonymously.**

***Keywords*– Stability, Frequencies, Transformation and Three Tooth Milling Process**

I. INTRODUCTION

Chatter is applied in this work to mean regenerative machine tool vibration. Regenerative vibration of milling process is governed by periodic delay differential equation. Stability analysis of such a system normally follows from model transformation from infinite dimensional phase space of periodic delay differential equation to finite dimensional phase space of an ordinary differential equation or discrete map that has stability characteristics that closely approximates that of the original system. It is of interest to know the type of loss stability occurring in a machining operation because this knowledge could of use in validation of a stability result of the operation. This type of validation will be of good use in stability analysis of milling process since it is a hybrid of analytical and numerical methods. Loss of stability in turning process is a bifurcation of Hopf type which has been proven experimentally by Shi and Tobias [1] and analytically by Stepan and Kalmar-Nagy [2] to have subcritical nature. This means that chatter frequencies (stability boundary frequencies) are related to unstable periodic motions about the stable stationary cutting [1]. It has been proven both experimentally and analytically that milling bifurcation is either of secondary Hopf or flip type [1] thus curve of marginal stability on the parameter plane of milling process could be adjudged valid if it could be demonstrated to exhibit these two types of bifurcation. The objective here to numerically conduct this type of validation for a three tooth end milling process and partition the stability chart into portions of secondary Hopf and flip bifurcation.

II. MODEL TRANSFORMATION

The general linear periodic delay-differential equation model for milling process is

$$\ddot{z} + 2\xi\omega_n\dot{z} + \left(\omega_n^2 + \frac{wh(t)}{m}\right)z = \frac{wh(t)}{m}z_\tau \dots\dots\dots\dots (1)$$

The following compact notations are used in equation (1); $z(t) = z$ and $z(t-\tau) = z_\tau$. With the substitutions $y_1 = z \; and \; y_2 = \dot{z}$ made, equation (1) could be put in state differential equation form as

$$\begin{Bmatrix}\dot{y}_1\\ \dot{y}_2\end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\left(\omega_n^2 + \dfrac{wh(t)}{m}\right) & -2\xi\omega_n\end{bmatrix}\begin{Bmatrix}y_1\\ y_2\end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ \dfrac{wh(t)}{m} & 0\end{bmatrix}\begin{Bmatrix}y_{1,\tau}\\ y_{2,\tau}\end{Bmatrix} \dots\dots (2)$$

Where $y_{i,\tau} = y_i(t-\tau)$ for $= 1 \; and \; 2$. The details of generation of equations (1) and (2) are seen in literature [1, 3]

The quantity $h(t)$ is called the specific force variation for the system which is τ −periodic where $\tau = \frac{60}{N\Omega}$. The symbol Ω is the spindle speed and N is the number of teeth of the end miller. Milling is thus a delayed Mathieu system. Specific force variation was derived for a three tooth end milling to have the form [4]:

$$h(t) = C\gamma(v\tau)^{\gamma-1}\left\{\frac{1}{2}\left\{1 + sgn\left[\sin\left(\left(\frac{\pi\Omega}{30}\right)t\right)\right]\right\}sin^\gamma\left(\left(\frac{\pi\Omega}{30}\right)t\right)\left[0.3sin\left(\left(\frac{\pi\Omega}{30}\right)t\right) + cos\left(\left(\frac{\pi\Omega}{30}\right)t\right)\right]\right.$$
$$+ \frac{1}{2}\left\{1 + sgn\left[\sin\left(\left(\frac{\pi\Omega}{30}\right)t + \frac{2\pi}{3}\right)\right]\right\}sin^\gamma\left(\left(\frac{\pi\Omega}{30}\right)t + \frac{2\pi}{3}\right)\left[0.3sin\left(\left(\frac{\pi\Omega}{30}\right)t + \frac{2\pi}{3}\right) + cos\left(\left(\frac{\pi\Omega}{30}\right)t + \frac{2\pi}{3}\right)\right]$$
$$+ \frac{1}{2}\left\{1 + sgn\left[\sin\left(\left(\frac{\pi\Omega}{30}\right)t + \frac{4\pi}{3}\right)\right]\right\}sin^\gamma\left(\left(\frac{\pi\Omega}{30}\right)t + \frac{4\pi}{3}\right)\left[0.3sin\left(\left(\frac{\pi\Omega}{30}\right)t + \frac{4\pi}{3}\right)\right.$$
$$\left.\left. + cos\left(\left(\frac{\pi\Omega}{30}\right)t + \frac{4\pi}{3}\right)\right]\right\} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (3)$$

In equation (3) C is workpiece material cutting coefficient, γ is the cutting force feed exponent, v is the prescribed feed speed and t is cutting duration from initial feed. For the reference system having the parameters; cutting coefficient $= 3.5 \times 10^{7}\mathrm{Nm}^{\frac{-7}{4}}$, $\gamma = 0.75$, feed speed $v = \frac{150mm}{min} = 0.0025m/s$, $N = 3$ and spindle speed $\Omega = 1000\mathrm{rpm}$, the graphical portrayal of equation (3) becomes:

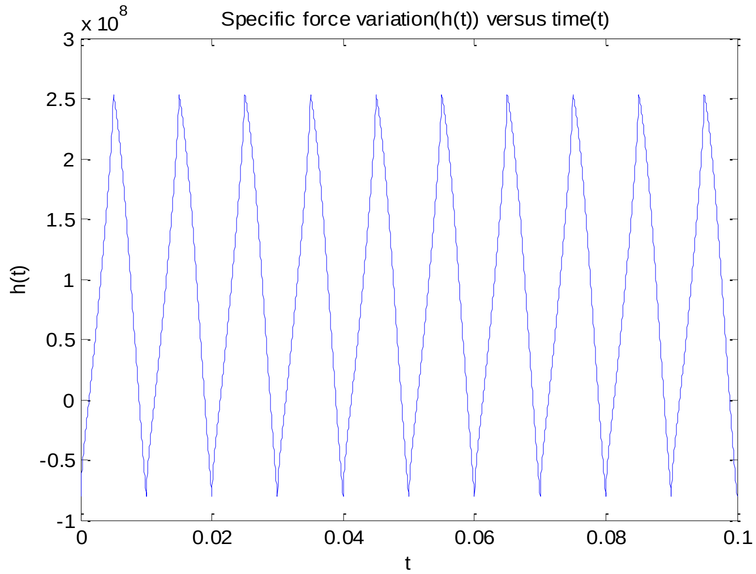

Figure1: Specific force variation

The Fargue-type approximation of periodic delay-differential equation (DDE) with periodic ordinary differential equation (ODE) utilizes the weight function [1].

$$w(\vartheta) = (-1)^n \frac{n^{n+1}}{\tau^{n+1} n!} \vartheta^n e^{n\vartheta/\tau} \ldots\ldots\ldots\ldots\ldots\ldots. (4)$$

Where parameter n can be any of $1, 2, 3, 4 \ldots \ldots$. The resulting ODE is investigated based on the Floquet theory for stability analysis of periodic ODE. Fargue proved the equality [1]

$$\lim_{n\to\infty} \int_{-\infty}^{0} w(\vartheta) z(t+\vartheta) d\vartheta = z(t-\tau) = z_\tau \ldots\ldots\ldots\ldots\ldots. (5)$$

From which the Fargue approximation for finite parameter n reads

$$z_\tau = z(t-\tau) \approx \int_{-\infty}^{0} w(\vartheta) z(t+\vartheta) d\vartheta \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Assuming that the Fargue approximation parameter n is big enough such equality becomes admissible in equation (6), substitution of equation (6) into equation (1) gives

$$\ddot{z} + 2\xi\omega_n \dot{z} + \left(\omega_n^2 + \frac{wh(t)}{m}\right) z = \frac{wh(t)}{m} \int_{-\infty}^{0} w(\vartheta) z(t+\vartheta) d\vartheta \ldots\ldots\ldots\ldots (7)$$

New variables of form

$$y_r = \int_{-\infty}^{0} \left[\frac{n!}{[n-(r-3)]!} \frac{w(\vartheta)}{\vartheta^{r-3}}\right] z(t+\vartheta) d\vartheta \ldots\ldots\ldots\ldots\ldots (8)$$

where $r = 3, 4, 5, \ldots\ldots (n+3)$, enables the formation of derivatives of form

$$\dot{y}_r = -\frac{n}{\tau} y_r - y_{r+1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

Putting equation (9) into equation (2) gives the state differential equation of milling process as

$$\begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \\ \dot{y}_5 \\ \vdots \\ \vdots \\ \dot{y}_{n+2} \\ \dot{y}_{n+3} \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ -\left(\omega_n^2 + \frac{wh(t)}{m}\right) & -2\xi\omega_n & \frac{wh(t)}{m} & 0 & 0 & \dots & 0 \\ 0 & 0 & -\frac{n}{\tau} & -1 & 0 & \dots & 0 \\ 0 & 0 & 0 & -\frac{n}{\tau} & -1 & \dots & 0 \\ 0 & 0 & 0 & 0 & -\frac{n}{\tau} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -\frac{n}{\tau} & -1 \\ (-1)^n \frac{n^{n+1}}{\tau^{n+1}} & 0 & 0 & 0 & \dots & 0 & -\frac{n}{\tau} \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ \vdots \\ \vdots \\ y_{n+2} \\ y_{n+3} \end{Bmatrix} \dots\dots\dots (10)$$

This is a periodic ODE having Stability characteristics that approaches that of equation (1) as n approaches infinity. The solution of equation (10) has the form

$$\boldsymbol{y}(t) = \boldsymbol{\Phi}(\boldsymbol{t})\boldsymbol{y}(0) \dots\dots\dots\dots\dots\dots\dots (11)$$

Where $\boldsymbol{\Phi}(\boldsymbol{t})$ is the fundamental matrix of the system which has been proven by Floquet to have the form [1]

$$\boldsymbol{\Phi}(t) = \mathbf{P}(t)e^{\boldsymbol{B}t} \dots\dots\dots\dots\dots\dots\dots (12)$$

Where $\boldsymbol{B}$ is a constant square matrix. $\mathbf{P}(t)$ has the following properties; periodicity such that if T is the principal period then $\mathbf{P}(t) = \mathbf{P}(t+T)$ and Initial condition of identity matrix such that $\mathbf{P}(0) = \boldsymbol{I}$. These two properties imply that

$$\boldsymbol{\Phi}(T) = e^{\boldsymbol{B}T} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (13)$$

$e^{\boldsymbol{B}T}$ is called the *principal* or *monodromy* or *Floquet transition matrix*. At time of one period after the initial condition equation (11) becomes

$$\boldsymbol{y}(T) = \boldsymbol{\Phi}(T)\boldsymbol{y}(0) \dots\dots\dots\dots\dots\dots\dots (14)$$

In equation (14) the monodromy matrix could be considered a second rank tensor that transforms the initial state $\boldsymbol{y}(0)$ to the state $\boldsymbol{y}(T)$ a principal period later in the same vector space. Equation (14) represents the discrete time map of the system (1), stability of which approximates the stability of the considered milling process. This stability result is obtained by investigating the nature of eigenvalues of the monodromy matrix in the cutting parameter space.

The very large magnitude of the element $A_{n+3,1} = (-1)^n \frac{n^{n+1}}{\tau^{n+1}}$ of the coefficient matrix at Fargue approximation parameter n large enough for close similarity of stability characteristics between equation (1) and (10) causes ill-conditioning of the latter. This type of numerical problem could be avoided by making use of the dimensionless time $\tilde{t} = \frac{n}{\tau}t$ [1]. If note is made of the fact that the period of the dimensionless time $\tilde{t}$ is the approximation parameter n then equation (1) under this change of variables becomes

$$\left(\frac{n}{\tau}\right)^2 \frac{d^2z}{d\tilde{t}^2} + 2\xi\omega_n\left(\frac{n}{\tau}\right)\frac{dz}{d\tilde{t}} + \left(\omega_n^2 + \frac{wh(\tilde{t})}{m}\right)z = \frac{wh(\tilde{t})}{m}z_n \dots\dots\dots\dots (15)$$

Where a similar compact notation as utilized in equation (1) is utilized in equation (15) as $z = z(\tilde{t})$ *and* $z_n = z(\tilde{t} - n)$. Re-arranging equation (15) to have the modal form gives

$$\frac{d^2z}{d\tilde{t}^2} + 2\xi\omega_n\left(\frac{\tau}{n}\right)\frac{dz}{d\tilde{t}} + \left(\omega_n^2 + \frac{wh(\tilde{t})}{m}\right)\left(\frac{\tau}{n}\right)^2 z = \frac{wh(\tilde{t})}{m}\left(\frac{\tau}{n}\right)^2 z_n \dots\dots\dots\dots (16)$$

A comparison of equations (1) and (16) gives that the ODE equivalent of equation (16) is

$$\begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \\ \dot{y}_5 \\ \vdots \\ \vdots \\ \dot{y}_{n+2} \\ \dot{y}_{n+3} \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ -\left(\frac{\tau}{n}\right)^2\left(\omega_n^2 + \frac{wh(\tilde{t})}{m}\right) & -2\xi\omega_n\left(\frac{\tau}{n}\right) & \frac{wh(\tilde{t})}{m}\left(\frac{\tau}{n}\right)^2 & 0 & 0 & \dots & 0 \\ 0 & 0 & -1 & -1 & 0 & \dots & 0 \\ 0 & 0 & 0 & -1 & -1 & \dots & 0 \\ 0 & 0 & 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 & -1 \\ (-1)^n & 0 & 0 & 0 & \dots & 0 & -1 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ \vdots \\ \vdots \\ y_{n+2} \\ y_{n+3} \end{Bmatrix} \dots\dots (17)$$

It is seen that the coefficient matrix of equation (17) is not likely to present numerical problems thus becoming the basis of the following stability analysis of the milling process.

Stability condition

From the Floquet theory the *fundamental matrix* is already given to have the form

$$\boldsymbol{\Phi}(t) = \mathrm{P}(t)e^{\boldsymbol{B}t} \dots\dots\dots\dots\dots\dots\dots (12)$$

By the similarity transformation of second rank tensors, the fundamental matrix could be written in the form

$$\boldsymbol{\Phi}(t) = \mathrm{P}(t)e^{(\boldsymbol{VDV}^{-1})t} \dots\dots\dots\dots\dots\dots\dots (18)$$

Where it is assumed that the eigen-vectors of $\boldsymbol{B}$ are linearly independent such that $\boldsymbol{V}$ is the matrix of eigen-vectors and $\boldsymbol{D}$ is a diagonal matrix with eigen-values of $\boldsymbol{B}$ as elements. The fundamental matrix is a matrix exponential function

$$\boldsymbol{\Phi}(t) = \mathrm{P}(t)\left[\boldsymbol{I} + (\boldsymbol{VDV}^{-1})t + \frac{1}{2!}(\boldsymbol{VDV}^{-1})^2t^2 + \frac{1}{3!}(\boldsymbol{VDV}^{-1})^3t^3 + \cdots\right] \dots\dots\dots.. (19)$$

$$\boldsymbol{\Phi}(t) = \mathrm{P}(t)\left[\boldsymbol{V}\left(\boldsymbol{I} + \boldsymbol{D}t + \frac{1}{2!}\boldsymbol{D}^2t^2 + \frac{1}{3!}\boldsymbol{D}^3t^3 + \cdots\right)\boldsymbol{V}^{-1}\right] \dots\dots\dots.. (20)$$

It is clear from equation (20) that the fundamental matrix becomes

$$\boldsymbol{\Phi}(t) = \mathrm{P}(t)\boldsymbol{V}e^{\boldsymbol{D}t}\boldsymbol{V}^{-1} \dots\dots\dots\dots.. (21)$$

The eigenvalues of $\boldsymbol{B}$, called *characteristic exponents* are designated λ_i. Equation (21) is put into equation (11) to give

$$\boldsymbol{y}(t) = \mathrm{P}(t)\boldsymbol{V}e^{\boldsymbol{D}t}\boldsymbol{V}^{-1}\boldsymbol{y}(0) \dots\dots\dots\dots\dots\dots\dots\dots.. (22)$$

This is written in expanded form thus

$$\boldsymbol{y}(t) = \mathrm{P}(t)\boldsymbol{V}\begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & e^{\lambda_3 t} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & e^{\lambda_4 t} & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & e^{\lambda_5 t} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & e^{\lambda_{n+2} t} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & e^{\lambda_{n+3} t} \end{bmatrix}\boldsymbol{V}^{-1}\boldsymbol{y}(0) \dots\dots. (23)$$

From equation (23) it is clear that the condition for asymptotic stability of the milling process is that each of the characteristic exponents has a negative real part. The monodromy matrix is seen from equation (23) to have the form

$$\boldsymbol{\Phi}(T)=\boldsymbol{V}\begin{bmatrix} e^{\lambda_1 T} & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & e^{\lambda_2 T} & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & e^{\lambda_3 T} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & e^{\lambda_4 T} & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & e^{\lambda_5 T} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & e^{\lambda_{n+2} T} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & e^{\lambda_{n+3} T} \end{bmatrix}\boldsymbol{V}^{-1} \dots\dots (25)$$

The eigenvalues of the monodromy matrix $\boldsymbol{\Phi}(T)$, designated μ_i are called the *characteristic multipliers*. Since the eigenvectors of monodromy matrix $\boldsymbol{\Phi}(T)$ are identical with those of $\boldsymbol{B}$, similarity transformation enables the transformation of equation (25) into

$$\begin{bmatrix} \mu_1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & \mu_2 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \mu_3 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \mu_4 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \mu_5 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \mu_{n+2} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \mu_{n+3} \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 T} & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & e^{\lambda_2 T} & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & e^{\lambda_3 T} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & e^{\lambda_4 T} & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & e^{\lambda_5 T} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & e^{\lambda_{n+2} T} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & e^{\lambda_{n+3} T} \end{bmatrix} \dots\dots (26)$$

From equation (26) it can be seen that the relationship between characteristic *multipliers* μ_i and *characteristic exponents* λ_i is

$$\mu_i = e^{\lambda_i T} \dots\dots\dots\dots\dots\dots\dots.. (27)$$

If a characteristic exponent is given as $\lambda_i = \sigma + j\omega$ then $\mu_i = e^{\sigma T} e^{j\omega T}$. It follows that

$$|\mu_i| = e^{\sigma T} \dots\dots\dots\dots\dots\dots (28)$$

It is already stated that stability of equation (1) requires all characteristic exponents to have negative real parts, that is $\sigma < 0$. From equation (28) the stability criterion for the system can also be stated to mean that all characteristic multipliers have modulus less than one, that is$|\mu_i| < 1$. This means that stability of the system requires all characteristic multipliers to exist inside a unit circle centred at the origin of the complex plane. It can be implied from equation (28) that at critical operating conditions when there exist characteristic multipliers such that $|\mu_i| = 1$ that the corresponding characteristic exponents are pure imaginary. This is a bifurcation condition that occurs when a least one characteristic multiplier is moving out of the unit circle centred on the origin of the complex plane. There are three possibilities [1, 5] as shown on figure2;

i. Period one bifurcation in which a characteristic multiplier leaves the unit circle at $+1$.
ii. Period two or period doubling or flip bifurcation in which the exit of the unit circle of the critical characteristic multiplier is at -1.
iii. Secondary Hopf or Neimark-Sacker bifurcation which involves a pair of complex conjugate characteristic multiplier leaving the unit circle.

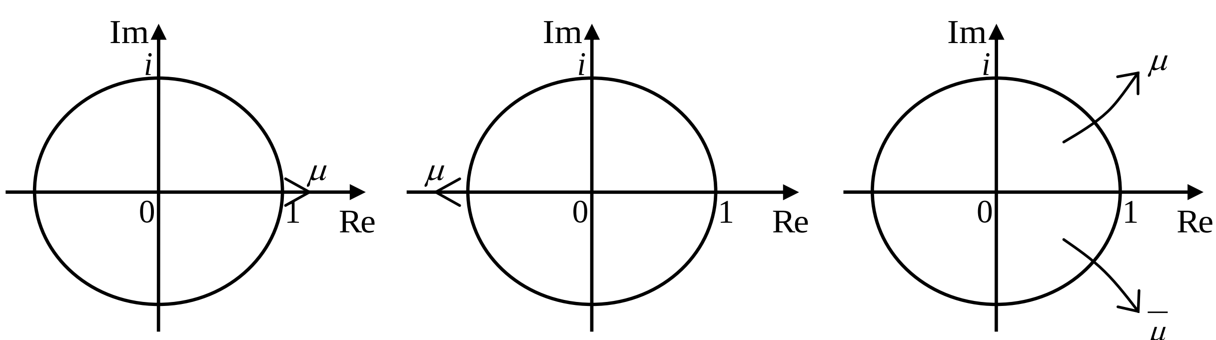

Figure2: a. Period one bifurcation b. Flip bifurcation c. Secondary Hopf bifurcation

Analytical generation of Floquet fundamental matrix is difficult thus numerical approximations of the monodromy matrix are used in stability analysis of periodic ODE's. A typical method of achieving an estimate of the principal matrix is by piecewise constant approximation of the of the time-varying coefficient matrix $\boldsymbol{A}(t)$ [1]. This involves dividing the principal period T of the system into k time intervals $[t_i, t_{i+1}]$ where $i = 0, 1, 2, \dots\dots\dots (k-1)$ and approximating the coefficient matrix of the system $\boldsymbol{A}(t)$ at the midpoint of each of the time intervals. The exponential matrix of the approximate coefficient matrices are coupled in multiplicative sense. In this work equal time intervals $\Delta t = t_{i+1} - t_i$ are used such that an approximate piecewise constant matrix for the system based on equation (17) is

$$\boldsymbol{A}\left(t_i+\frac{\Delta t}{2}\right)=\boldsymbol{A}_i=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ -\left(\frac{\tau}{n}\right)^2\left(\omega_n^2+\frac{wh\left(t_i+\frac{\Delta t}{2}\right)}{m}\right) & -2\xi\omega_n\left(\frac{\tau}{n}\right) & \frac{wh\left(t_i+\frac{\Delta t}{2}\right)}{m}\left(\frac{\tau}{n}\right)^2 & 0 & 0 & \dots & 0 \\ 0 & 0 & -1 & -1 & 0 & \dots & 0 \\ 0 & 0 & 0 & -1 & -1 & \dots & 0 \\ 0 & 0 & 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 & -1 \\ (-1)^n & 0 & 0 & 0 & \dots & 0 & -1 \end{bmatrix} \dots (29)$$

The estimation of the Floquet transition matrix utilizing equal time intervals becomes

$$\boldsymbol{\Phi}\left(\frac{n}{\tau}\mathrm{T}\right)=\prod_{i=0}^{k-1} e^{A_i\frac{n}{\tau}\Delta t} \dots\dots\dots\dots\dots\dots\dots\dots\dots (30)$$

Stability curve plotted using the equations (29 an 30) is the locus of points on the plane of the cutting parameters at which maximum magnitude of the eigenvalues is one. Using equation (30), the achieved stability chart at Fargue approximation parameter $n = 325$ and discretization integer $k = 10$ of principal period of one discrete delay τ for an end milling machine with parameters $m = 0.0431kg, \omega_n = 5700rad/sec\ and\ \xi = 0.02\ and\ C = 3.5 \times 10^7 \mathrm{Nm}^{\frac{-7}{4}}$ is shown in figure3.

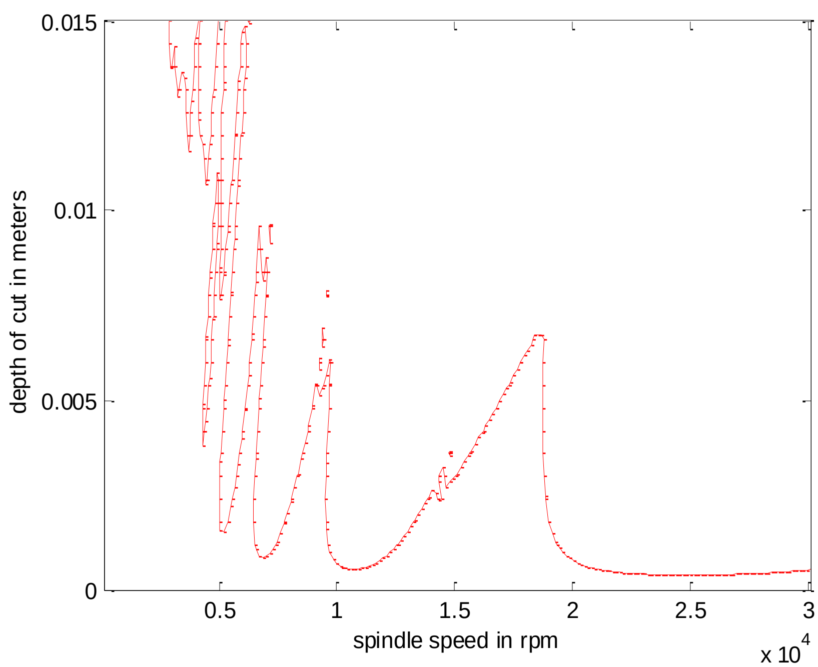

Figure3: Stability chart of the reference system

Milling stability boundary frequencies

For a pair of complex conjugate characteristic exponents$(\lambda, \bar{\lambda})$, a solution of form $z(t) = p(t)e^{\lambda t} + \bar{p}(t)e^{\bar{\lambda} t}$ exists for the milling process [1]. This is expected of a linear equation, where $p(t)$ and $\bar{p}(t)$ are τ −periodic. For the case of period one bifurcation λ=0 thus $z(t) = p(t) + \bar{p}(t)$ and $z(t+\tau) = p(t+\tau) + \bar{p}(t+\tau)$=$p(t) + \bar{p}(t)$ meaning that $z(t) = z(t+\tau)$. Putting this result into equation (1) gives that characteristic multiplier leaving the unit circle at $+1$ results in a damped oscillator $\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n^2 z = 0$ which by Routh-Hurwitz criterion is asymptotically stable for $\xi > 0$. Period one bifurcation is thus excluded as a form of loss of stability for the linear milling process. Period two bifurcation occurring means that $\mu = -1 = e^{j(\pi + k2\pi)} = e^{j\omega_{p2}\tau}, k = 0, 1, 2 \ldots\ldots and\ j = \sqrt{-1}$. Thus;

$$\omega_{p2} = \frac{\pi}{\tau} + k\frac{2\pi}{\tau} = \frac{N\Omega\pi}{60} + k\frac{N\Omega\pi}{30} \ldots\ldots\ldots (31)$$

The number of teeth of the milling tool N is 3. Equation (31) gives that there are infinitely many stability boundary frequencies stemming from period two bifurcation. The subscript '$p2$' is used on the period two bifurcation chatter frequency to differentiate it from other types of stability boundary frequencies. Stability boundary frequencies arising from secondary Hopf bifurcation are extracted from Fourier analysis of the equation $z(t) = p(t)e^{\lambda t} + \bar{p}(t)e^{\bar{\lambda} t}$[1]. Under this type of critical condition, $z(t) = p(t)e^{j\omega t} + \bar{p}(t)e^{-j\omega t}$ results. By Fourier analysis of the periodic function $p(t)$ the results obtained are [6];

$$p(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$
$$c_k = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} p(t)e^{-jk\omega_0 t}\, dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (32)$$

where $\omega_0 = \frac{2\pi}{\tau}$ is the fundamental natural frequency of the system. Use made of the result of Fourier analysis of $p(t)$ gives a solution of form

$$z(t) = \sum_{k=-\infty}^{\infty}\left[c_k e^{j\left(\omega + k\frac{2\pi}{\tau}\right)t} + \bar{c}_k e^{-j\left(-\omega + k\frac{2\pi}{\tau}\right)t}\right] \ldots\ldots\ldots (33)$$

It becomes clear that the infinitely many secondary Hopf bifurcation stability boundary frequencies are

$$\omega_{sh} = \pm\omega + k\frac{2\pi}{\tau} = \pm\omega + k\frac{n\pi\Omega}{30} \ldots\ldots\ldots\ldots\ldots\ldots (34)$$

A more detailed attention is then given to Fig. 3 in order to find out which bifurcation different portions of the stability boundary represent. This can be ascertained by substituting boundary parameter combinations into the monodromy matrix of the system as given in equation (30) and calculating the maximum magnitude characteristic multipliers. Results of some of such substitutions are as given the table below:

Table1: Critical characteristic multipliers

Ω *in rpm*	*w in mm*	*critical eigenvalues* ω
25000	0.5	-0.0200 - 1.0251i -0.0200 + 1.0251i
17000	5.0	-0.9709
18700	2.5	0.9704 - 0.1435i 0.9704 + 0.1435i
15000	3	-1.0237
12100	1	-0.7882 - 0.6515i -0.7882 + 0.6515i
9450	5.5	-1.0141
7000	1	-0.2426 + 1.0106i -0.2426 - 1.0106i

The slight deviation of magnitude of critical eigenvalues from one results from error of reading boundary parameter combination from figure3. It can be seen from table1 that in conformity with theory, only secondary Hopf and flip bifurcation are possible in the three tooth milling process. It has to be recalled that Secondary Hopf or Neimark-Sacker bifurcation involves a pair of complex conjugate characteristic multipliers leaving the unit circle centred at the origin of the complex plane while Period two or period doubling or flip bifurcation involve the exit of the unit circle of the critical characteristic multiplier at -1. Using the results of table1 and many other such numerical results not shown, stability boundary of figure3 is demarcated using solid vertical lines into portions of Secondary Hopf (SH) bifurcation and flip (F) bifurcation as shown on figure4. The portions of Secondary Hopf (SH) bifurcation contain spindle speeds that are found in equation (34) while those of flip (F) bifurcation contain spindle speeds that are found in equation (31).

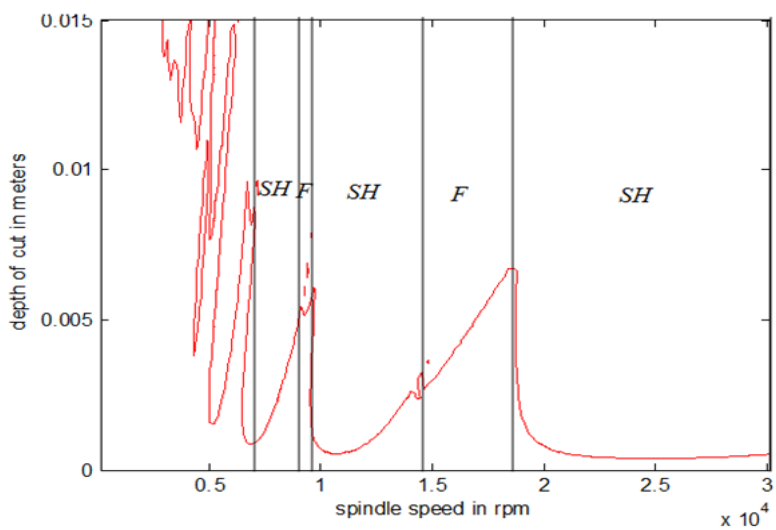

Figure4: Stability chart showing domains of Seconday Hopf (SH)and Flip(F) Bifurcations

III. CONCLUSION

Generalized stability condition of periodic systems with discrete delay is established via similarity transformation of second order tensors. The details of generation of stability chart of a three tooth end milling process with the parameters; $m = 0.0431kg, \omega_n = 5700rad/sec\ and\ \xi = 0.02\ and\ C = 3.5\times 10^7\text{Nm}^{\frac{-7}{4}}$ was outlined. By substitution of critical milling parameter combination into the monodromy matrix it was demonstrated that the critical characteristic multipliers of the system are of the secondary Hopf and flip type. This was seen to conform to theory and practice thus validating the stability chart for the system.

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