Solutions of the Anisotropic Problem of the EPDE Using BIEM

Hassan Ghassemi[1,*], Abdulkhaled Zareei[2] and Alireza Kazemipour[3]
[1]Deaprtment of Maritime Engineering, Amirkabir University of Technology, Tehran, Iran
[2,3]School of Engineering, Persian Gulf University, Bushehr, Iran
[1,*]gasemi@aut.ac.ir

***Abstract*– This paper presents to solve an elliptic partial differential equation (EPDE) using boundary integral equation method (BIEM/BEM). This equation is employed to the anisotropic problem. First, this equation should be transformed into the canonical form in order to obtain the Green's functions. Then, having the Green's function, the integral representation of the solution is obtained by applying Green's identity for any point on the boundary or in the domain.**

***Keywords*– Canonical Form, Green's Function, Anisotropic and Boundary Integral Equation Method/Boundary Element Method (BIEM/BEM)**

I. INTRODUCTION

In mathematics, a PDE is a differential equation that contains unknown multivariable functions and their partial derivatives. PDEs are used to formulate problems involving functions of several variables, and are either solved by hand, or used to create a relevant computer model.

PDEs can be used to describe a wide variety of phenomena such as sound, heat, electrostatics, electrodynamics, fluid flow, elasticity, or quantum mechanics. These seemingly distinct physical phenomena can be formalized similarly in terms of PDEs. Just as ordinary differential equations often model one-dimensional dynamical systems, partial differential equations often model multidimensional systems. PDEs find their generalization in stochastic partial differential equations [1].

The method of characteristics is a technique for solving PDEs. Typically, it applies to first-order equations, although more generally the method of characteristics is valid for any hyperbolic partial differential equation. The method is to reduce a PDE to a family of ODEs along which the solution can be integrated from some initial data given on a suitable hypersurface [2].

One of the main problems in the analysis of acoustic vibrations in internal cavities is to determine the natural frequencies of the acoustic system and its associated vibration modes. In the present work develops through the BEM formulation and modal analysis in two-dimensional geometries to compare the results with the FEM modal analysis based on the DR-BEM which is self-developed method provides a good approximation compared with the results obtained by FEM. Predicting noise and its control is becoming increasingly important in the design of a variety of systems [3].

This paper describes the canonical form of the PDE in order to obtain Green's function and the boundary integral equation form. This integral equation form can be applied to the boundary of body to determine the quantity of the problem.

II. MATHEMATICAL EQUATION

The theory of partial differential equations of the second order is a great deal more complicated than that of the equations of the first order, and it is much more typical of the subject as a whole. Within the context, considerably better results can be achieved for equations of the second order in two independent variables than for equations in space of higher dimensions. Linear equations are the easiest to handle. A second order linear partial differential equation for the *anisotropic* problems is defined as [4]:

$$Au_{xx} + Bu_{xy} + Cu_{yy} = f(x,y),\ in\ \Omega \qquad (1)$$

Subject to the mixed boundary conditions as:

$$\begin{cases} u = \bar{u} & on\ \Gamma_1 \\ \nabla u.\vec{m} = \bar{u}_n & on\ \Gamma_2 \end{cases} \qquad (2)$$

Where,

$$\begin{aligned} \vec{m} &= m_x i + m_y j \\ &= (An_x + Bn_y)i + (Bn_x + Cn_y)j \end{aligned} \qquad (3)$$

In case of $B=0$ and $A=C=1$, $\vec{m} = \vec{n} = n_x i + n_y j$.

$\vec{m}$ is a vector in the direction of the connormal to the boundary. A, C and B are constants and satisfying the ellipticity condition $B^2 - 4AC < 0$.

Boundary condition (2) and equation (3) can be arranged as follows:

$$\nabla u.\vec{m} = q.n = q_n \qquad (4)$$

Where,

$$\vec{q} = q_x i + q_y j = (Au_x + Bu_y) i + (Bu_x + Cu_y) j \quad (5)$$

$\vec{q}$ is a vector representing the *flux* of the *u*, which is the flow of the potential *u* through the boundary Γ per unit length of the boundary, and q_n its component in the direction of the normal to the boundary (Fig. 1).

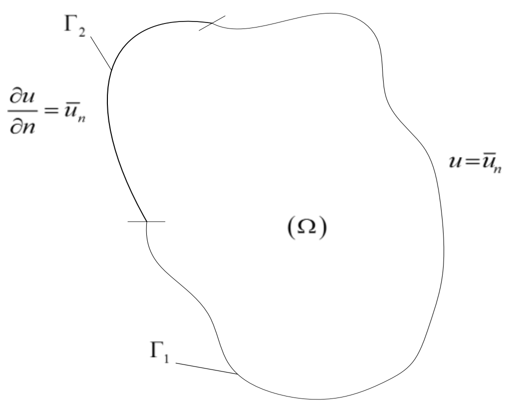

Fig. 1. Domain Ω and boundary $\Gamma = \Gamma_1 \bigcup \Gamma_2$

III. CANONICAL FORM

Using characteristic equations, we are obtained:

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{B + \sqrt{B^2 - 4AC}}{2A} \\ \dfrac{dy}{dx} = \dfrac{B - \sqrt{B^2 - 4AC}}{2A} \end{cases} \quad (6)$$

The integral of equations (6) is called the characteristic curves. Since the equations are first-order ordinary differential equations (ODEs), the solutions may be written as:

$$\begin{cases} y = \lambda_1 x + c_1, \\ y = \lambda_2 x + c_2 \end{cases} \quad (7)$$

where λ_1 and λ_2 are complex numbers. Accordingly, c_1 and c_2 are allowed to take on complex values. Thus,

$$\begin{cases} \xi = y - (\lambda_1) x \\ \eta = y - (\lambda_2) x, \end{cases} \quad (8)$$

where $\lambda_{1,2} = a \pm ib$ in which *a* and *b* are real constants, and

$$\begin{cases} a = \dfrac{B}{2A}, \quad b = \dfrac{\sqrt{|D|}}{2A} \\ where: |D| = \begin{vmatrix} 2A & B \\ B & 2C \end{vmatrix} = 4AC - B^2 \end{cases} \quad (9)$$

Since ξ and η are complex, we introduce new real variables

$$\begin{cases} \alpha = \dfrac{1}{2}(\xi + \eta) = y - ax, \\ \beta = \dfrac{1}{2i}(\xi - \eta) = -bx \end{cases} \quad (10)$$

With these variables, we transform the u_{xx}, u_{yy} and u_{xy} into the new variables

$$\begin{cases} u_x = u_\alpha \alpha_x + u_\beta \beta_x \\ u_y = u_\alpha \alpha_y + u_\beta \beta_y \\ u_{xx} = u_{\alpha\alpha} \alpha_x^2 + 2u_{\alpha\beta} \alpha_x \beta_x \\ \qquad + u_{\beta\beta} \beta_x^2 + u_\alpha \alpha_{xx} + u_\beta \beta_{xx} \\ u_{xy} = u_{\alpha\alpha} \alpha_x \alpha_y + u_{\alpha\beta} (\alpha_x \beta_y + \alpha_y \beta_x) \\ \qquad + u_{\beta\beta} \beta_x \beta_y + u_\alpha \alpha_{xy} + u_\beta \beta_{xy} \\ u_{yy} = u_{\alpha\alpha} \alpha_y^2 + 2u_{\alpha\beta} \alpha_y \beta_y \\ \qquad + u_{\beta\beta} \beta_y^2 + u_\alpha \alpha_{yy} + u_\beta \beta_{yy} \end{cases} \quad (11)$$

Application of this transformation readily reduces equation (1) to the canonical form

$$\begin{cases} u_{\alpha\alpha} + u_{\beta\beta} = \dfrac{1}{|D|} f(\alpha, \beta) \\ or \\ u_{\alpha\alpha} + u_{\beta\beta} = \dfrac{1}{|D|} f(\alpha, \beta, u, u_\alpha, u_\beta) \end{cases} \quad (12)$$

This is canonical form. When *A*, *B* and *C* are constants so $D=4AC-B^2$ is constant as well. f is independent/dependent of *u* or its derivative, therefore it is linear or nonlinear PDE. Eq. (12) is a poison equation that can be solved even analytical or numerical. Here, we give *two examples* for changing into the canonical form.

Example 1: Change the PDE $u_{xx} + 2u_{xy} + 5u_{yy} = x + y$ into canonical form.
Here $A = 1$, $B = 1$ and $C = 5$, and $B^2 - AC = -16$ so the PDE is elliptic. The characteristic Equation (6) gives $dy/dx = 1 \pm i$. Thus $y = (1 \pm 2i)x$, and we set

$\xi = y-(1+2i)x$ and $\eta = y-(1-2i)x$. Going through the usual chain rule, we find u_{xx}, u_{xy} and u_{yy} and adding into reference equation, we find $u_{\xi\eta}=0$, of course the PDE has complex numbers present. To eliminate them we make another change of variables. We set

$$\begin{cases} \alpha = \frac{1}{2}(\xi+\eta) = y-x, \\ \beta = \frac{1}{2i}(\xi-\eta) = -2x \end{cases}$$

Then, applying the chain rule again, we find

$$u_{\alpha\alpha}+u_{\beta\beta}=\alpha-\beta$$

with α and β real variables. This is a Poison equation and can be solved analytically.

Example 2: Change the PDE $u_{xx}+2u_{xy}+5u_{yy}+u_x = x+y$ into canonical form. Doing the same process, we find

$$u_{\alpha\alpha}+u_{\beta\beta}=\frac{1}{4}(u_\alpha-2u_\beta)+\alpha-\beta$$

This is a nonlinear PDE. It is not easy to find a general solution of this equation; therefore it should be solved through numerical method such as BEM.#

IV. GREEN'S FUNCTION

In order to employ the BEM, Green's function is required. To find Green's function, we should find the fundamental solution of the PDE. First the governing equation of the PDE should be transformed into the canonical form, and then Green's function can be obtained.

Consider a point source placed at point $P(x,y)$ of the *xy*-plane (Figure 2). Its density at $Q(\xi,\eta)$ may be expressed mathematically by the delta function as $f(Q)=\delta(Q-P)$ and the potential $G=G(Q,P)$ produced at point *Q* satisfies the Laplace's equation

$$\nabla^2 G = \delta(Q-P) \tag{13}$$

A free space Green's functions (*G*) is determined by:

$$G=\frac{1}{2\pi}\ln r, \quad where: r=|Q-P| \tag{14}$$

Green's function for the examples 1 and 2 can be determined by its canonical forms that is given by:

$$G=\frac{1}{2\pi\sqrt{|D|}}\ln r \tag{15}$$

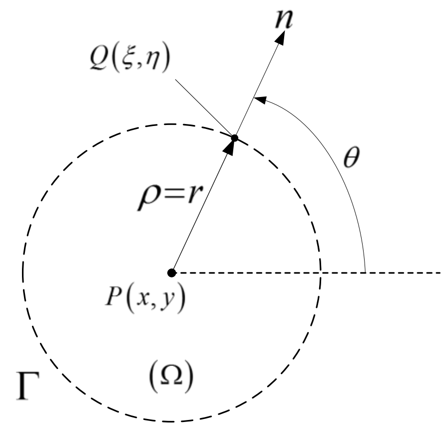

Fig. 2. Circular domain Ω of radius *r* with a source *P* at its center

V. SOLUTION OF BIEM

The solution of the Laplace's equation $(\nabla^2 u=0)$ in the boundary integral equation (BIE) for any point *P* can be written as:

$$e(p)u(p) = -\int_\Gamma G(p,q)\frac{\partial u(q)}{\partial n_q}ds_q + \int_\Gamma u(q)\frac{\partial G(p,q)}{\partial n_q}ds_q \tag{16}$$

Where,

$$e(p)=\begin{cases} 1 & for\ P\ inside\ \Omega \\ \frac{\theta}{2\pi} & for\ P\in\Gamma \\ 0 & for\ P\ outside\ \Omega \end{cases} \tag{17}$$

and $\frac{\partial G}{\partial n}=G_n=\frac{1}{2\pi}\frac{\vec{r}.\vec{n}}{r^2}$

θ is the angle between the tangents of the boundary at point *P*. for smooth boundary $\theta=\pi$, then $e(p)=0.5$.

For the *Poison equation* $(\nabla^2 u = f)$, the BIE for point *p* on the smooth boundary is expressed as:

$$0.5u=-\int_\Gamma Gu_n ds+\int_\Gamma uG_n ds+\int_\Omega Gfds \tag{18}$$

In the process of solving the Poisson equation by BEM, domain integrals appear in the integral representation of the solution (18). This integral is of the form $\int_\Omega Gfds$.

Although the integrand Gf is known, the fact that domain integrals need to be evaluated spoils the pure boundary character of the method, thus weakening the advantages of BEM over domain methods. However, it is possible to overcome this drawback by employing the Dual Reciprocity

Method (DRM) which was first introduced by Nardini and Brebbia [5]. They used the method to establish a consistent mass matrix in an effort to solve dynamic problems by utilizing the BEM and the static fundamental solution. Since then, the DRM was further developed to solve elliptic problems for which the fundamental solution could not be determined or was difficult to treat numerically, as well as to solve parabolic and hyperbolic problems employing simple static fundamental solutions. *Nonlinear* problems have also been attacked by this method.
The next problem we obtain the BIE for anisotropic equation, i.e. Eq (1) like example 1.

Equation of PDE: $u_{xx} + 2u_{xy} + 5u_{yy} = x + y$

Canonical form: $\nabla^2 u = \alpha - \beta$
where: $\alpha = y - x, \beta = -2x$

BIE solution is obtained from the Eq (18):

$$0.5u = -\int_\Gamma G u_n ds + \int_\Gamma u G_n ds + \int_\Omega (x+y) G ds \quad (19)$$

Here, $G = \frac{1}{2\pi\sqrt{|D|}} \ln r, \quad G_n = \frac{1}{2\pi\sqrt{|D|}} \frac{\vec{r}.\vec{n}}{r^2}$

Eq. (19) is boundary integral form of the anisotropic problem.

VI. CONCLUSIONS

The task of solving a class of EPDE that arises frequently in the formulation of engineering problems involving non-homogeneous anisotropic media is considered. With using the characteristics equations, the PDE is transformed into a canonical form that allows finding the Green's function and to be formulated in terms of an integral equation suitable for the development of a BEM.

ACKNOWLEDGMENT

The authors would like to acknowledge the assistance provided by the Marine Research Center (MRC) in the form of a research grant for carrying out the work reported.

REFERENCES

[1]. Myint-U T., Debnath L., Linear Partial Differential Equations for Scientists and Engineers, Fourth edition, 2007 Birkhauser Boston.
[2]. Miersemann E., Partial Differential Equations, lecture notes, Department of Mathematics, Leipzig university, Oct 2012.
[3]. Zakerdoost, H., Ghassemi , H., Dual Reciprocity Boundary Element Method for Steady State Convection-Diffusion-Radiation problems, International Journal of Partial Differential Equations and Applications. 2014, 2(4), 68-71.
[4]. Katsikadelis, J. T., Boundary element, theory and application, Elsevier publication, 2002.
[5]. Nardini, D. and Brebbia, C.A., 1982. New Approach to Vibration Analysis Using Boundary Elements, in Boundary Element Methods in Engineering, Computational Mechanics Publications, Southampton and Springer-Verlag, Berlin.