Transitivity Action of A_n (n= 5,6,7) on Unordered and Ordered Triples

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Abstract– **In this paper, we study some transitivity action of the alternating group A_n (n = 5,6,7) acting on unordered triples and ordered triples from the set X = { 1,2,3,---,n } through determination of the number of disjoint equivalence classes called orbits. When n ≤ 7, the alternating group acts transitively on both $X^{(3)}$ and $X^{[3]}$.**

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I. PRELIMINARIES

A) Notation and Terminology

In this paper, we shall represent the following notations as: $\sum_i$ - sum over i; S_n- Symmetric group of degree n and order n!; A_n – an alternating group of degree n and order n!/2; $|G|$ -The order of a group G; $|G:H|$ -Index of H in G; $X^{(3)}$ – The set of an unordered triples from the set X = { 1,2,3,---,n }; $X^{[3]}$ – The set of an ordered triple from set X = { 1,2,3,---,n }; {a,b,c}-An unordered triple; [a,b,c] – An ordered triple.

We also define some basic terminologies on permutation group and give some results on group actions as:

Definition 1.1.1

Let X be a set. A group G acts on the left of X if for each $g \in G$ and each $x \in X$ there corresponds a unique element $gx \in X$ such that:

i) $(g_1g_2) = g_1(g_2x)$, $g_1,g_2 \in G$ and $x \in X$.

ii) For any $x \in X$, 1x=x, where 1 is the identity in G.

The action of G from the right on X can be defined in a similar way. In fact it is merely a matter of taste whether one writes the group elements on the left or on the right.

Definition 1.1.2

Let G act on a set X. Then X is partitioned into disjoint equivalent classes called orbits or transitivity classes of the action. For each $x \in X$ the orbit containing x is called the orbit of x and is denoted by $orb_G(x)$.

Definition 1.1.3

Let G act on a set X and let $x \in X$. The stabilizer of x in G, denoted by $stab_G(x)$ is given by $stab_G(x) = \{g \in G \mid gx=x\}$.

Note: $stab_G(x)$ forms a subgroup of G which is also called the isotropy group of X. This subgroup is also denoted by G_x.

Definition 1.1.4

Let G act on a set X. The set of elements of X fixed by $g \in G$ is called the fixed point set of g and is denoted by Fix(g). Thus $Fix(g)=\{x \in X \mid gx=x\}$.

Definition 1.1.5

If the action of a group G on a set X has only one orbit, then we say that G acts transitively on X. In other words, G acts transitively on X if for every pair of points $x,y \in X$, there exists $g \in G$ such that gx=y.

Definition 1.1.6

Let X be a non-empty set. A permutation of X is a one-to-one mapping of X onto itself.

Theorem 1.1.7 [Orbit-Stabilizer Theorem] [Rose 1978]

Let G act on a set X and let $x \in X$, then $|orb_G(x)| = |G : stab_G(x)|$.

Theorem 1.1.8 [Cauchy-Frobenius Lemma] [Harary 1969]

Let G be a finite group acting on a set X. The number of orbits of G is:

$$\frac{1}{|G|}\sum_{g\in G}|Fix(g)|.$$

This theorem is usually but erroneously attributed to Burnside (1911) cf. Neumann (1977).

Theorem 1.1.9 [Krishnamurthy 1985]

Two permutations in A_n are conjugate if and only if they have the same cycle type, and if $g \in G$ has cycle type $(\alpha_1,\alpha_2,\ldots,\alpha_n)$, then the number of permutations in A_n conjugate to g is:

$$\frac{n!}{\prod_{i=1}^{n} \alpha_i ! i^{\alpha_i}}.$$

Definition 1.1.10

Let X be the set {1,2,---,n}, then the symmetric group of degree n is the group of all the permutations of X under the binary operation of composition of maps. It is denoted by S_n and has order n!.

Definition 1.1.11

The subgroup of S_n consisting of all the even permutations is called the alternating group. The group is denoted by A_n.

The order of A_n is $\frac{n!}{2}$.

Definition 1.1.12

If a finite group G acts on a set X with n elements, each $g \in G$ corresponds to a permutation δ of X, which can be written uniquely as a product of disjoint cycles. If δ has α_1 cycles of length1, α_2 cycles of length 2,---,α_n cycles of length n, then we say that δ and hence g has cycle type $(\alpha_1,\alpha_2,---,\alpha_n)$.

B) Introduction

In 1970, Higman computed the rank and the subdegrees of the symmetric group S_n acting on pairs from the set X= {1, 2, 3,… ,n}. He found out that the rank is three and the subdegrees are; 1, 2(n-2), $\binom{n-2}{2}$.

In 1972, Cameron [1] worked on suborbits of multiply transitive permutation groups and later in 1974, he studied suborbits of primitive groups. In 1999, Rosen [5] dealt with the properties arising from the action of a group on unordered and ordered pairs. Based on these results, we investigate some properties of the action of A_n on $X^{(3)}$ the set of all unordered triples from the set X = {1,2,3,---,n} and on $X^{[3]}$ the set of all ordered triples from the set X = {1,2,3,---,n}.

The alternating group A_n acts on the set $X^{(3)}$ by the rule, $g\{x,y,z\}=\{gx,gy,gz\}\ \forall\ g\in A_n$ and $\{x,y,z\} \in X^{(3)}$.

II. ACTION OF THE ALTERNATING GROUP A_n ON UNORDERED TRIPLES

A) Some general results of permutation groups acting on $X^{(3)}$

We first give the proofs of two lemmas which will be very useful in the investigation of the action of A_n on $X^{(3)}$

Lemma 2.1.1: Let G be the alternating group A_n acting on the set X = {1,2,3,---,n} and $g \epsilon G$ have cycle type $(\alpha_1,\alpha_2,---,\alpha_n)$. Then the number of elements in $X^{(3)}$ fixed by g is given by the formula:

$$|\mathrm{Fix}(g)| = \binom{\alpha_1}{3} + \alpha_2\alpha_1+\alpha_3, \text{ where } \alpha_1 \geq 3.$$

Proof:

Let $g \in A_n$ have cycle type $(\alpha_1,\alpha_2,---,\alpha_n)$. Then $\{a,b,c\} \in X^{(3)}$ is fixed by g if each of a,b and c come from a single cycle in g or one of a, b or c come from a single cycle in g and the other two come from a 2-cycle in g or a, b, c come from a 3-cycle in g. From the first case, the number of unordered triples fixed by g is $\binom{\alpha_1}{3}$.

From the second case, the number of unordered triples fixed by g is $\alpha_2\alpha_1$ and in the third case the number of unordered triples fixed by g is α_3. Therefore, the number of unordered triples fixed by:
g is

$$\binom{\alpha_1}{3} + \alpha_2\alpha_1+\alpha_3.$$

Lemma 2.1.2: Let G be the alternating group A_n acting on the set X = {1,2,3,---,n} and $g \epsilon G$ have cycle type $(\alpha_1,\alpha_2,---,\alpha_n)$. Then the number of permutations in A_n fixing $\{a,b,c\} \in X^{(3)}$ and having the same cycle type as g is given by:

$$\frac{(n-3)!}{(\alpha_1-3)!\,1^{\alpha_1-3}\prod_{i=2}^{n}\alpha_i!\,i^{\alpha_i}} + \frac{3(n-3)!}{(\alpha_1-1)!\,1^{\alpha_1-1}(\alpha_2-1)!\,2^{\alpha_2-1}\prod_{i=3}^{n}\alpha_i!\,i^{\alpha_i}} + \frac{2(n-3)!}{\alpha_1!\,1^{\alpha_1}\,\alpha_2!\,2^{\alpha_2}(\alpha_3-1)!\,3^{\alpha_3-1}\prod_{i=4}^{n}\alpha_i!\,i^{\alpha_i}}$$

where $\alpha_1 \geq 3$, $\alpha_2 \geq 1$, $\alpha_3 \geq 1$.

Proof:

A permutation $g \in A_n$ fixes an unordered triple say $\{a,b,c\} \in X^{(3)}$ as in the following cases;

Case1:

If g maps each element a, b and c onto itself, that is each of the elements a, b and c comes from a single cycle. To get the number of permutations in A_n that fix {a, b, c} and having the same cycle type as g, we apply Theorem 1.1.9 to a permutation in A_{n-3} with cycle type $(\alpha_{1-3},\alpha_2,\alpha_3,---,\alpha_n)$ to get

$$\frac{(n-3)!}{(\alpha_1-3)!\,1^{\alpha_1-3}\prod_{i=2}^{n}\alpha_i!\,i^{\alpha_i}}$$

Case 2:

If one of the elements a, b and c comes from a single cycle and the other two come from a 2-cycle. In this case, a, b and c may come from any of the following three permutations; (ab) (c) ---, (ac) (b) --- or (bc) (a) ---. Applying Theorem 1.1.9 to a permutation of A_{n-3} with cycle type $(\alpha_1-1,\alpha_2-1,\alpha_3,\alpha_4,---,\alpha_n)$, we get,

$$\frac{(n-3)!}{(\alpha_1-1)!\,1^{\alpha_1-1}\,(\alpha_2-1)!\,2^{\alpha_2-1}\prod_{i=3}^{n}\alpha_i!\,i^{\alpha_i}}$$

Considering the three cases above, we get in total

$$\frac{3\,(n-3)!}{(\alpha_1-1)!\,1^{\alpha_1-1}\,(\alpha_2-1)!\,2^{\alpha_2-1}\prod_{i=3}^{n}\alpha_i!\,i^{\alpha_i}}$$

Case 3:

If the elements a, b and c come from a 3-cycle in g. In this case a, b and c may come from the permutation (abc) or (acb). Applying Theorem 1.1.9 to a permutation of A_{n-3} with cycle type (α_1, α_2, α_3-1, α_4,---,α_n), we get,

$$\frac{(n-3)!}{\alpha_1!\,1^{\alpha_1}\,\alpha_2!\,2^{\alpha_2}\,(\alpha_3-1)!\,3^{\alpha_3-1}\prod_{i=4}^{n}\alpha_i!\,i^{\alpha_i}}$$

Combining the two cases above, we get

$$\frac{2(n-3)!}{\alpha_1!\,1^{\alpha_1}\,\alpha_2!\,2^{\alpha_2}\,(\alpha_3-1)!\,3^{\alpha_3-1}\prod_{i=4}^{n}\alpha_i!\,i^{\alpha_i}}$$ permutations.

Combining cases (1), (2) and (3) we get the required result.

B) Some properties of the action of G = A_5 on $X^{(3)}$

The alternating group A_5 is a subgroup of S_5 containing all even permutations and has order $|A_5| = \frac{5!}{2} = 60$.

The cardinality of $X^{(3)}$ is $\binom{5}{3} = 10$.

Theorem 2.2.1

A_5 acts transitively on $X^{(3)}$.

Proof :

First, we use the Cauchy – Frobenius Lemma (Theorem 1.1.8). By Definition 1.1.5, it suffices to show that the action of A_5 on $X^{(3)}$ has only one orbit. Let the cycle type of $g \in A_5$ be (α_1,α_2,---,α_n). Then the number of permutations in A_5 with the same cycle type as g is given by:

Theorem 1.1.9. The number of elements in $X^{(3)}$ fixed by each $g \in A_5$ is given by Lemma 2.1.1. We now have the following Table 2.2.1.

Table 2.2.1: Permutations in A_5 and the number of fixed points

Permutation type	Cycle type	Number of permutations	|Fix(g)| in X(3)
1	(5,0,0,0,0)	1	10
(abc)	(2,0,1,0,0)	20	1
(abcde)	(0,0,0,0,1)	24	0
(ab) (cd)	(1,2,0,0,0)	15	2

We now apply the Cauchy- Frobenius Lemma (Theorem 1.1.8) to get the number of orbits of A_5 on $X^{(3)}$

$$\frac{1}{|A_5|}\sum_{g\in G}|\mathrm{Fix}(g)| = \frac{1}{60}[(1 \text{ x } 10) + (20 \text{ x } 1) + (24 \text{ x } 0) + (15 \text{ x } 2) = 1.$$

Thus A_5 acts transitively on $X^{(3)}$.

We can also show that A_5 acts transitively on $X^{(3)}$ using the Orbit-Stabilizer Theorem (Theorem 1.1.7). Here it is enough to show that the cardinalities of the orbit of a triple say $\{1,2,3\} \in X^{(3)}$ and the set $X^{(3)}$ are equal, which implies that the action of G = A_5 on $X^{(3)}$ has only one orbit. Using Lemma 2.1.2 we have the following Table 2.2.2.

Table 2.2.2: Number of permutations in $stab_G$ {1,2,3}

Permutation type	Cycle type	Number of permutations in $stab_G$ {1,2,3}
1	(5,0,0,0,0)	1
(abc)	(2,0,1,0,0)	2
(abcde)	(0,0,0,0,1)	0
(ab) (cd)	(1,2,0,0,0)	3

From the third column, we see that:

$|\mathrm{Stab}_G\{1,2,3\}| = 1 + 2 + 0 + 3 = 6.$

Therefore, by Orbit-Stabilizer Theorem, we have that

$|\mathrm{Orb}_G\{1,2,3\}| = [G: \mathrm{Stab}_G\{1,2,3\}].$

$$= \frac{|G|}{|\mathrm{Stab}_G\{1,2,3\}|} = \frac{60}{6} = 10$$

$$= |X^{(3)}|$$

Hence A_5 acts transitively on $X^{(3)}$.

C) Some properties of the action of G = A_6 on $X^{(3)}$

The alternating group A_6 is a subgroup of S_6 containing all even permutations and has order $|A_6| = \frac{6!}{2} = 360.$

The cardinality of $X^{(3)}$ is $\binom{6}{3} = 20.$

Theorem 2.3.1

A_6 acts transitively on $X^{(3)}$.

Proof:

First, we use the Cauchy – Frobenius Lemma (Theorem 1.1.8). By Definition 1.1.5, it suffices to show that the action of A_6 on $X^{(3)}$ has only one orbit. Let the cycle type of $g \in A_6$ be $(\alpha_1, \alpha_2, ---, \alpha_n)$. Then the number of permutations in A_6 with the same cycle type as g is given by Theorem 1.1.9. The number of elements in $X^{(3)}$ fixed by each $g \in A_6$ is given by Lemma 2.1.1. We now have the following Table 2.3.1.

Table 2.3.1: Permutations in A_6 and the number of fixed points

Permutation type	Cycle type	Number of permutations	$\|Fix(g)\|$ in $X^{(3)}$
1	(6,0,0,0,0,0)	1	20
(abc)	(3,0,1,0,0,0)	40	2
(abcde)	(1,0,0,0,1,0)	144	0
(ab) (cd)	(2,2,0,0,0,0)	45	4
(abc) (def)	(0,0,2,0,0,0)	40	2
(ab) (cdef)	(0,1,0,1,0,0)	90	0

We now apply the Cauchy- Frobenius Lemma (Theorem 1.1.8) to get the number of orbits of A_6 on $X^{(3)}$

$$\frac{1}{|G|} \sum_{g \in G} |Fix(g)| = \frac{1}{360} [(1 \times 20) + (40 \times 2) + (144 \times 0) + (45 \times 4) + (40 \times 2) + (90 \times 0)] = 1.$$

$|G|$

Thus A_6 acts transitively on $X^{(3)}$.

We can also show that A_6 acts transitively on $X^{(3)}$ using the Orbit-Stabilizer Theorem (Theorem 1.1.7). Here it is enough to show that the cardinalities of the orbit of a triple say $\{1,2,3\} \in X^{(3)}$ and the set $X^{(3)}$ are equal, which implies that the action of $G = A_6$ on $X^{(3)}$ has only one orbit. Using Lemma 2.1.2 we have the following Table 2.3.2.

Table 2.3.2: Number of permutations in $stab_G$ {1,2,3}

Permutation type	Cycle type	Number of permutations in $stab_G$ {1,2,3}
1	(6,0,0,0,0,0)	1
(abc)	(3,0,1,0,0,0)	4
(abcde)	(1,0,0,0,1,0)	0
(ab) (cd)	(2,2,0,0,0,0)	9
(abc) (def)	(0,0,2,0,0,0)	4
(ab) (cdef)	(0,1,0,1,0,0)	0

From the third column, we see that

$|Stab_G\{1,2,3\}| = 1 + 4 + 0 + 9 + 4 + 0 = 18$

Therefore, by Orbit-Stabilizer Theorem, we have that

$|Orb_G\{1,2,3\}| = [G: Stab_G\{1,2,3\}].$

$$= \frac{|G|}{|Stab_G\{1,2,3\}|} = \frac{360}{18} = 20 = |X^{(3)}|.$$

Hence A_6 acts transitively on $X^{(3)}$.

□

D) Some properties of the action of G = A_7 on $X^{(3)}$

The alternating group A_7 is a subgroup of S_7 containing all even permutations and has order $|A_7| = \frac{7!}{2} = 2520.$

The cardinality of $X^{(3)}$ is $\binom{7}{3} = 35.$

Theorem 2.4.1

A_7 acts transitively on $X^{(3)}$.

Proof:

First, we use the Cauchy – Frobenius Lemma (Theorem 1.1.8). By Definition 1.1.5, it suffices to show that the action of A_7 on $X^{(3)}$ has only one orbit. Let the cycle type of $g \in A_7$ be $(\alpha_1, \alpha_2, ---, \alpha_n)$. Then the number of permutations in A_7 with the same cycle type as g is given by Theorem 1.1.9. The number of elements in $X^{(3)}$ fixed by each $g \in A_7$ is given by Lemma 2.1.1. We now have the following Table 2.4.1.

Table 2.4.1: Permutations in A_7 and the number of fixed points

Permutation type	Cycle type	Number of permutations	$\|Fix(g)\|$ in $X^{(3)}$
1	(7,0,0,0,0,0,0)	1	35
(abc)	(4,0,1,0,0,0,0)	70	5
(abcde)	(2,0,0,0,1,0,0)	504	0
(abcdefg)	(0,0,0,0,0,0,1)	720	0
(ab) (cd)	(3,2,0,0,0,0,0)	105	7
(ab) (cd) (efg)	(0,2,1,0,0,0,0)	210	1
(abc) (def)	(1,0,2,0,0,0,0)	280	2
(ab) (cdef)	(1,1,0,1,0,0,0)	630	1

We now apply the Cauchy- Frobenius Lemma (Theorem 1.1.8) to get the number of orbits of A_7 on $X^{(3)}$

$$\frac{1}{|A_7|} \sum_{g \in G} |Fix(g)|$$

$= \frac{1}{2520}$ [(1X35)+(70X5)+(504X0)+(720X0)+(105X7)+(210 X1)+(280X2)+(630X1)] = 1.

Thus A_7 acts transitively on $X^{(3)}$.

We can also show that A_7 acts transitively on $X^{(3)}$ using the Orbit-Stabilizer Theorem (Theorem 1.1.7). Here it is enough to show that the cardinalities of the orbit of a triple say $\{1,2,3\} \in X^{(3)}$ and the set $X^{(3)}$ are equal, which implies that the action of $G = A_7$ on $X^{(3)}$ has only one orbit. Using Lemma 2.1.2 we have the following Table 2.4.2.

Table 2.4.2: Number of permutations in $stab_G$ {1,2,3}

Permutation type	Cycle type	Number of permutations in $stab_G$ {1,2,3}
1	(7,0,0,0,0,0,0)	1
(abc)	(4,0,1,0,0,0,0)	10
(abcde)	(2,0,0,0,1,0,0)	0
(abcdefg)	(0,0,0,0,0,0,1)	0
(ab) (cd)	(3,2,0,0,0,0,0)	21
(ab) (cd) (efg)	(0,2,1,0,0,0,0)	6
(abc) (def)	(1,0,2,0,0,0,0)	16
(ab) (cdef)	(1,1,0,1,0,0,0)	18

From the third column, we see that

$|Stab_G\{1,2,3\}| = 1 + 10 + 0 + 0 + 21 + 6 + 16 + 18 = 72$

Therefore, by Orbit-Stabilizer Theorem, we have that

$|Orb_G\{1,2,3\}| = [G: Stab_G\{1,2,3\}]$.

$$= \frac{|G|}{|Stab_G\{1,2,3\}|} = \frac{2520}{72} = 35 = |X^{(3)}|.$$

Hence A_7 acts transitively on $X^{(3)}$.

E) Some general results of permutation groups acting on $X^{[3]}$

Similarly like in section 2.1 we give the proofs of two lemmas which will be very useful in the investigation of the transitivity of A_n on $X^{[3]}$.

Lemma 3.1.1: Let G be the alternating group A_n acting on the set X = {1,2,3,---,n} and g∈G have cycle type $(\alpha_1,\alpha_2,---,\alpha_n)$. Then the number of elements in $X^{[3]}$ fixed by g is given by the formula

$$|Fix(g)| = 3!\binom{\alpha_1}{3}, \text{ where } \alpha_1 \geq 3.$$

Proof:

Let $[a, b, c] \in X^{[3]}$ and $g \in A_n$ .Then g fixes[a, b, c] if and only if g[a,b,c]=[g(a),g(b),g(c)]=[a,b,c].

Which implies that g(a)=a, g(b)=b, g(c)=c. Thus each of the elements a,b and c comes from 1-cycles. Therefore, the number of unordered triples fixed by $g \in A_n$ is $\binom{\alpha_1}{3}$.

Further, an unordered triple say, {a,b,c} can be rearranged to give 3! different ordered triples.

Hence the number of ordered triples fixed by $g \in A_n$ is $3!\binom{\alpha_1}{3}$.

Lemma 3.1.2: Let G be the alternating group A_n acting on the set X = {1,2,3,---,n} and g∈G have cycle type $(\alpha_1,\alpha_2,---,\alpha_n)$. Then the number of permutations in A_n fixing $[a,b,c] \in X^{[3]}$ and having the same cycle type as g is given by:

$$\frac{(n-3)!}{(\alpha_1-3)!\, 1^{\alpha_1-3} \prod_{i=2}^{n} \alpha_i!\, i^{\alpha_i}}$$

where $\alpha_1 \geq 3$.

Proof:

Let $g \in A_n$ have cycle type $(\alpha_1,\alpha_2,---,\alpha_n)$. Then g fixes an ordered triples [a,b,c] if and only if each of the elements a,b and c come from a single cycle. The number of permutations in A_n having the same cycle type as g and fixing [a,b,c] is equal to the number of permutations in A_{n-3} having cycle type $(\alpha_1-3,\alpha_2,---,\alpha_n)$. By Theorem 1.1.9, this number is:

$$\frac{(n-3)!}{(\alpha_1-3)!\, 1^{\alpha_1-3} \prod_{i=2}^{n} \alpha_i!\, i^{\alpha_i}}$$

F) Some properties of the action of $G = A_5$ on $X^{[3]}$

The cardinality of $X^{[3]}$ is $3!\binom{5}{3} = 60$.

Theorem 3.2.1: A_5 acts transitively on $X^{[3]}$

Proof:

First, we use the Cauchy – Frobenius Lemma (Theorem 1.1.8). By Definition 1.1.5, it suffices to show that the action of A_5 on $X^{[3]}$ has only one orbit. Let the cycle type of $g \in A_5$ be $(\alpha_1,\alpha_2,---,\alpha_n)$. Then the number of permutations in A_5 with the same cycle type as g is given by Theorem 1.1.9. The number of elements in $X^{[3]}$ fixed by each $g \in A_5$ is given by Lemma 3.1.1. We now have the following Table 3.2.1.

Table 3.2.1: Permutations in A_5 and the number of fixed points

Permutation type	Cycle type	Number of permutations	$\lvert Fix(g) \rvert$ in $X^{[3]}$
1	(5,0,0,0,0)	1	60
(abc)	(2,0,1,0,0)	20	0
(abcde)	(0,0,0,0,1)	24	0
(ab) (cd)	(1,2,0,0,0)	15	0

We now apply the Cauchy- Frobenius Lemma (Theorem 1.1.8) to get the number of orbits of A_5 on $X^{[3]}$

$$\frac{1}{\lvert A_5 \rvert} \sum_{g \in G} \lvert Fix(g) \rvert = \frac{1}{60}[(1 \text{ x } 60) + (20 \text{ x } 0) + (24 \text{ x } 0) + (15 \text{ x } 0)] = 1.$$

Thus A_5 acts transitively on $X^{[3]}$.

We can also show that A_5 acts transitively on $X^{[3]}$ using the Orbit-Stabilizer Theorem (Theorem 1.1.7). Here it is enough to show that the cardinalities of the orbit of a triple say $[1,2,3] \in X^{[3]}$ and the set $X^{[3]}$ are equal, which implies that the action of $G = A_5$ on $X^{[3]}$ has only one orbit. Using Lemma 3.1.2 we have the following Table 3.2.2.

Table 3.2.2: Number of permutations in $stab_G$ [1,2,3]

Permutation type	Cycle type	Number of permutations in $stab_G$ [1,2,3]
1	(5,0,0,0,0)	1
(abc)	(2,0,1,0,0)	0
(abcde)	(0,0,0,0,1)	0
(ab) (cd)	(1,2,0,0,0)	0

From the third column, we see that

$\lvert Stab_G [1,2,3] \rvert = 1 + 0 + 0 + 0 = 1.$

Therefore, by Orbit-Stabilizer Theorem, we have that

$\lvert Orb_G [1,2,3] \rvert = [G: Stab_G [1,2,3]].$

$$= \frac{\lvert G \rvert}{\lvert Stab_G [1,2,3] \rvert} = \frac{60}{1} = 60$$

$$= \lvert X^{[3]} \rvert.$$

Hence A_5 acts transitively on $X^{[3]}$.

G) Some properties of the action of G = A_6 on $X^{[3]}$

The cardinality of $X^{[3]}$ is $3!\binom{6}{3} = 120$.

Theorem 3.3.1: A_6 acts transitively on $X^{[3]}$

Proof:

First, we use the Cauchy – Frobenius Lemma (Theorem 1.1.8). By Definition 1.1.5, it suffices to show that the action of A_6 on $X^{[3]}$ has only one orbit. Let the cycle type of $g \in A_6$ be $(\alpha_1, \alpha_2, \text{---}, \alpha_n)$. Then the number of permutations in A_6 with the same cycle type as g is given by Theorem 1.1.9. The number of elements in $X^{[3]}$ fixed by each $g \in A_6$ is given by Lemma 3.1.1. We now have the following Table 3.3.1.

Table 3.3.1: Permutations in A_6 and the number of fixed points

Permutation type	Cycle type	Number of permutations	$\lvert Fix(g) \rvert$ in $X^{[3]}$
1	(6,0,0,0,0,0)	1	120
(abc)	(3,0,1,0,0,0)	40	6
(abcde)	(1,0,0,0,1,0)	144	0
(ab) (cd)	(2,2,0,0,0,0)	45	0
(abc) (def)	(0,0,2,0,0,0)	40	0
(ab) (cdef)	(0,1,0,1,0,0)	90	0

We now apply the Cauchy- Frobenius Lemma (Theorem 1.1.8) to get the number of orbits of A_6 on $X^{[3]}$

$$\frac{1}{\lvert A_6 \rvert} \sum_{g \in G} \lvert Fix(g) \rvert = \frac{1}{360}[(1 \text{ x } 120) + (40 \text{ x } 6) + (144 \text{ x } 0) + (45 \text{ x } 0) + (40 \text{ x } 0) + (90 \text{ x } 0)] = 1.$$

Thus A_6 acts transitively on $X^{[3]}$.

We can also show that A_6 acts transitively on $X^{[3]}$ using the Orbit-Stabilizer Theorem (Theorem 1.1.7). Here it is enough to show that the cardinalities of the orbit of a triple say $[1,2,3] \in X^{[3]}$ and the set $X^{[3]}$ are equal, which implies that the action of $G = A_6$ on $X^{[3]}$ has only one orbit. Using Lemma 3.1.2 we have the following Table 3.3.2.

Table 3.3.2: Number of permutations in $stab_G$ [1,2,3]

Permutation type	Cycle type	Number of permutations in $stab_G$ [1,2,3]
1	(6,0,0,0,0,0)	1
(abc)	(3,0,1,0,0,0)	2
(abcde)	(1,0,0,0,1,0)	0
(ab) (cd)	(2,2,0,0,0,0)	0
(abc) (def)	(0,0,2,0,0,0)	0
(ab) (cdef)	(0,1,0,1,0,0)	0

From the third column, we see that

$\lvert Stab_G [1,2,3] \rvert = 1 + 2 + 0 + 0 + 0 + 0 = 3.$

Therefore, by Orbit-Stabilizer Theorem, we have that

$\lvert Orb_G [1,2,3] \rvert = [G: Stab_G [1,2,3]].$

$$= \frac{\lvert G \rvert}{\lvert Stab_G [1,2,3] \rvert} = \frac{360}{3} = 120$$

$$= \lvert X^{[3]} \rvert.$$

Hence A_6 acts transitively on $X^{[3]}$.

H) Some properties of the action of G = A_7 on $X^{[3]}$

The cardinality of $X^{[3]}$ is $3!\binom{7}{3} = 210$.

Theorem 3.4.1: A_7 acts transitively on $X^{[3]}$

Proof:

First, we use the Cauchy – Frobenius Lemma (Theorem 1.1.8). By Definition 1.1.5, it suffices to show that the action of A_7 on $X^{[3]}$ has only one orbit. Let the cycle type of $g \in A_7$ be $(\alpha_1,\alpha_2,---,\alpha_n)$. Then the number of permutations in A_7 with the same cycle type as g is given by Theorem 1.1.9. The number of elements in $X^{[3]}$ fixed by each $g \in A_7$ is given by Lemma 3.1.1. We now have the following Table 3.4.1.

Table 3.4.1: Permutations in A_7 and the number of fixed points

Permutation type	Cycle type	Number of permutations	$\lvert Fix(g) \rvert$ in $X^{[3]}$
1	(7,0,0,0,0,0,0)	1	210
(abc)	(4,0,1,0,0,0,0)	70	24
(abcde)	(2,0,0,0,1,0,0)	504	0
(abcdefg)	(0,0,0,0,0,0,1)	720	0
(ab) (cd)	(3,2,0,0,0,0,0)	105	6
(ab) (cd) (efg)	(0,2,1,0,0,0,0)	210	0
(abc) (def)	(1,0,2,0,0,0,0)	280	0
(ab) (cdef)	(1,1,0,1,0,0,0)	630	0

We now apply the Cauchy Frobenius Lemma (Theorem 1.1.8) to get the number of orbits of A_7 on $X^{[3]}$.

$$\frac{1}{\lvert A_7 \rvert}\sum_{g\in G} \lvert Fix(g) \rvert$$

$$= \frac{1}{2520}[(1X210)+(70X24)+(504X0)+(720X0)+(105X6)(210X0)+(280X0)+(630X0)]=1$$

Thus A_7 acts transitively on $X^{[3]}$.

We can also show that A_7 acts transitively on $X^{[3]}$ using the Orbit-Stabilizer Theorem (Theorem 1.1.7). Here it is enough to show that the cardinalities of the orbit of a triple say $[1,2,3] \in X^{[3]}$ and the set $X^{[3]}$ are equal, which implies that the action of G = A_7 on $X^{[3]}$ has only one orbit. Using Lemma 3.1.2 we have the following Table 3.4.2.

Table 3.4.2: Number of permutations in $stab_G$ [1,2,3]

Permutation type	Cycle type	Number of permutations in $stab_G$ [1,2,3]
1	(7,0,0,0,0,0,0)	1
(abc)	(4,0,1,0,0,0,0)	8
(abcde)	(2,0,0,0,1,0,0)	0
(abcdefg)	(0,0,0,0,0,0,1)	0
(ab) (cd)	(3,2,0,0,0,0,0)	3
(ab) (cd) (efg)	(0,2,1,0,0,0,0)	0
(abc) (def)	(1,0,2,0,0,0,0)	0
(ab) (cdef)	(1,1,0,1,0,0,0)	0

From the third column, we see that

$$\lvert Stab_G [1,2,3] \rvert = 1 + 8 + 0 + 0 + 3 + 0 + 0 + 0 = 12.$$

Therefore, by Orbit-Stabilizer Theorem, we have that

$$\lvert Orb_G [1,2,3] \rvert = [G: Stab_G [1,2,3]].$$

$$= \frac{\lvert G \rvert}{\lvert Stab_G [1,2,3] \rvert} = \frac{2520}{12} = 210 = \lvert X^{[3]} \rvert.$$

Hence A_7 acts transitively on $X^{[3]}$.

REFERENCES

[1]. Cameron, P. J. Permutation groups with multiply transitive suborbits I. Proc. *London Math*. Soc. 6: (1972), 136-140.

[2]. Higman, D. G. Finite permutation groups of rank 3. *Math. Zeitschrifl* 86: (1964), 145-156.

[3]. Krishnamurthy, V. *Combinatorics, theory and applications*. Affiliated East West PressPrivate Limited. New Delhi (1985).

[4]. Rose, J. S. *A Course in group theory*. Cambridge University Press. Cambridge (1978).

[5]. Rosen, K. H. *Handbook of Discrete and Combinational Mathematics*. CRC Press, New Jersey (1999).

[6]. Rotman, J. J. *The Theory of Groups*: An Introduction. Allyn and Bacon. Inc. Boston, USA (1973).

[7]. Wielandt, H. *Finite Permutation Groups*. Academic Press, New York and London (1964).