

# Solutions of the Acoustic Problem in the 3D Form of the Helmholtz Equation Using DRBEM

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**Abstract**– This paper presents to the solutions of the acoustic problem using dual reciprocity boundary element method (DRBEM). The acoustic propagates inside the silencer and its sound can be formulated by the three dimensional (3D) Helmholtz equation. The form of the equation is  $\nabla^2\phi + k^2\phi = b(\phi)$  which can be solved by the integral form on the boundary. The conventional boundary element method (BEM) is not suitable for solving the acoustic problems of silencers with higher Mach number subsonic flow, due to the presence of domain integral. The dual reciprocity method (DRM) is a method that converts the domain integral into the boundary integral. Mathematical formulations, discretization form and evaluation of the integrals are described and discussed. Algorithm of the method is also presented.

**Keywords**– Acoustic Problem, 3D Helmholtz Equation and Dual Reciprocity Method

## I. INTRODUCTION

Acoustic problem will influence the sound propagation inside the silencer system. In mathematical modeling of such problem, a three-dimensional Helmholtz equation can be employed to predict such a problem. The DRBEM method is an effective and powerful method, and has been developed to predict those problems [1].

Since the integral form of the equation has domain integral that is weak problem in the BEM. To change the domain integral into the boundary integral, Nardini and Brebbia was first introduced the DRM method [2]. After that, some researchers employed this method to solve the Poisson and Helmholtz equations. Ramachandran described this method on his textbook with full program and some examples [3]. Zakerdoost and Ghassemi presented this method to the steady state convection-diffusion-radiation problems [4]. Lee, et al [5] used the DRBEM to model the acoustic radiation in a subsonic non-uniform flow field, and indicated that the Sommerfeld-radiation condition at infinite was satisfied when DRBEM was used to deal with this problem.

In the real fluids, the sound wave will attenuate gradually as propagation due to the acoustic damping effect, and part of the acoustic energy is transformed into the heat energy. Dokumaci developed a boundary integral method, and applied it to solve radiation problems in viscous dissipation medium [6]. Kara and Ben Tahar applied the boundary integral method

to solve radiation problems in viscothermal dissipation medium [7].

The objectives of the present study are to derive the acoustic governing equation in three-dimensional potential flow with consideration of acoustic damping effect, and then to apply the boundary element method to predict the acoustic attenuation performance of acoustic problem in the silencer system with considerations of the 3D potential flow and acoustic damping effects.

The following sections are organized as follows. Section II is described the physical problem and obtain the governing equation. Section III is given the DRM method that how to convert the domain integral into the boundary integral. The evaluation integrals are described in section IV using numerical Gaussian quadrature method for regular integrals and analytical method for the singular integrals. Section V presents the algorithm of the DRBEM and finally conclusions are given in Section VI.

## II. PHYSICAL PROBLEM AND GOVERNING EQUATION

The complete dynamic basic equations in the fluid are the mass continuity, Navier-Stokes and energy equations [8]:

Mass continuity

$$\frac{\partial \rho}{\partial t} + \rho_0 \nabla \cdot u = 0. \quad (1)$$

Navier-Stokes

$$\rho_0 \frac{\partial u}{\partial t} = -\nabla p + \left(\frac{4}{3}\mu + \mu_v\right) \nabla(\nabla \cdot u). \quad (2)$$

Energy

$$\frac{dS}{dt} = 0. \quad (3)$$

State

$$p = c_0^2 \rho \quad (4)$$

where  $\rho_0$  and  $\rho$  are the ambient and excess density, respectively.  $u$  is the particle velocity,  $c_0$  and  $p$  and are

ambient sound velocity and sound pressure resulted from the acoustic disturbance, respectively;  $S$  is the entropy per unit mass;  $\mu$  and  $\mu_v$  are the coefficients of shear viscosity and volume viscosity, respectively.

Substituting Eq. (4) into Eq. (1) yields

$$\frac{1}{c_0^2} \frac{\partial p}{\partial t} + \rho_0 \nabla \cdot u = 0. \tag{5}$$

Substituting Eq. (5) into Eq. (1) yields

$$\rho_0 \frac{\partial u}{\partial t} = -\nabla p + \left(\frac{4}{3}\mu + \mu_v\right) \nabla \left(\frac{1}{\rho_0 c_0^2} \frac{\partial p}{\partial t}\right). \tag{6}$$

Differentiating Eq. (5) with respect to time  $t$  and Eq. (6) with respect to coordinates, and substituting yields

$$\nabla^2 p - \frac{1}{c_0^2} \frac{\partial^2 p}{\partial t^2} + \left(\frac{4}{3}\mu + \mu_v\right) \nabla^2 \left(\frac{1}{\rho_0 c_0^2} \frac{\partial p}{\partial t}\right) = 0. \tag{7}$$

Using total derivative  $\frac{d}{dt} = \frac{\partial}{\partial t} + V_0 \cdot \nabla$ , Eq. (7) can be written as

$$\begin{aligned} & \left[1 + \frac{1}{\rho_0 c_0^2} \left(\frac{4}{3}\mu + \mu_v\right) \frac{\partial}{\partial t}\right] \nabla^2 p - \frac{1}{c_0^2} \frac{\partial^2 p}{\partial t^2} \\ & - 2 \frac{\partial}{\partial t} (V_0 \cdot \nabla) p - \frac{1}{c_0^2} (V_0 \cdot \nabla)(V_0 \cdot \nabla) p \\ & + \frac{1}{\rho_0 c_0^2} \left(\frac{4}{3}\mu + \mu_v\right) \nabla^2 (V_0 \cdot \nabla) p = 0. \end{aligned} \tag{8}$$

The particle velocity  $u$  and the sound pressure  $p$  can be expressed in terms of the acoustic velocity potential  $\phi$  as

$$u = -\nabla \Phi \tag{9}$$

And pressure is expressed as

$$p = \rho_0 \left[ \frac{\partial \phi}{\partial t} + V_0 \cdot \nabla \phi - \left(\frac{4}{3}\mu + \mu_v\right) \nabla^2 \phi \right]. \tag{10}$$

Then, the Eq. (8) can be defined in terms of total velocity potential ( $\Phi$ ) as

$$\begin{aligned} & \left[1 + \frac{1}{\rho_0 c_0^2} \left(\frac{4}{3}\mu + \mu_v\right) \frac{\partial}{\partial t}\right] \nabla^2 \Phi - \frac{1}{c_0^2} \frac{\partial^2 \Phi}{\partial t^2} \\ & - 2 \frac{\partial}{\partial t} (M_0 \cdot \nabla \Phi) - \frac{1}{c_0^2} (M_0 \cdot \nabla)(M_0 \cdot \nabla \Phi) \\ & + \frac{1}{\rho_0 c_0^2} \left(\frac{4}{3}\mu + \mu_v\right) \nabla^2 (M_0 \cdot \nabla \Phi) = 0. \end{aligned} \tag{11}$$

where  $M_0 = V_0 / c_0$  is Mach number of the 3D potential flow.

For the steady state harmonic motion, the acoustic velocity motion can be expressed as

$$\Phi = \phi e^{j\omega t} = \phi(\cos \omega t + j \sin \omega t) \tag{12}$$

where  $\omega = 2\pi f$  is the angular frequency and  $f$  is the frequency and  $j = \sqrt{-1}$ .

Finally, the governing equation (11) can be written as

$$\begin{aligned} & \nabla^2 \phi + k^2 \phi = b(\phi), \text{ in } \Omega \\ & B.C. = \text{Known, on } \Gamma \end{aligned} \tag{13}$$

where:

$$\begin{cases} b(\phi) = \frac{k^2}{k_0^2} [2j(M_0 \cdot \nabla \phi) + (M_0 \cdot \nabla)(M_0 \cdot \nabla \phi)] \\ k = \frac{k_0}{\sqrt{1 + \frac{jk_0}{\rho_0 c_0} \left(\frac{4}{3}\mu + \mu_v\right)}} \\ k_0 = \omega / c_0 \end{cases} \tag{14}$$

$k_0$  is the wave number.

Eq. (13) is the governing equation for the sound field in the three-dimensional flow.

### III. DUAL RECIPROCITY METHOD (DRM)

In the boundary element method, for a body of the boundary  $\Gamma$  and domain  $\Omega$  (as shown in Figure 1), the integral formulation of the Eq. (13) may be expressed as

$$\begin{aligned} e(p)\phi(p) = & \int_{\Gamma} \left(G \frac{\partial \phi}{\partial n} - \phi \frac{\partial G}{\partial n}\right) ds \\ & - \int_{\Omega} G b(\phi) d\Omega \end{aligned} \tag{15}$$

where:

$$e(p) = \begin{cases} 1 & \text{for } P \text{ outside Surface} \\ 0.5 & \text{for } P \text{ on Surface} \\ 0 & \text{for } P \text{ inside Surface} \end{cases} \quad (16)$$

And  $G$  is the Green's function of the Helmholtz equation. For the 3D problem,

$$G = \frac{\exp(-jkr)}{4\pi r} \quad (17)$$

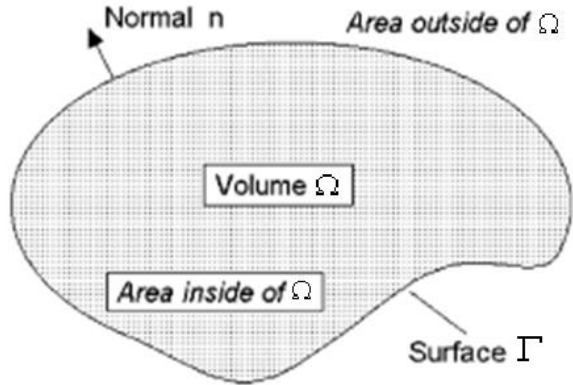


Fig. 1. Definition sketch of boundary domain

Eq. (13) contains the volume integral, which is difficult problem. Therefore, in order to overcome to this difficulty, one way may be converted to the boundary integral is the DRM [2-3]. This method is focused to the term  $b(\phi)$  which may be approximated by the following expression:

$$b(X) = \sum_{i=1}^{N+L} f_i \alpha_i \quad (18)$$

where  $\alpha_i, f_i$  are interpolation coefficients and radial basis function (RBF), respectively.  $N$  is the number of collocation nodes on the boundary,  $L$  is the number of collocation points in the domain and  $r_i$  is defined as the distance between the node under consideration and the node  $i$ .

For each simple source function  $f_i$  a particular solution  $h_i$  needs to be found and satisfied as

$$\nabla^2 h_i + k^2 h_i = f_i \quad (19)$$

One of the key ingredients of the DRM is the expression introduced in Eq. (18). There are many functions of the RBF to choose. It is usually chosen as

$$f_i = 1 + r_i \quad (20)$$

The particular solution of the Eq. (19) can be found as follows  $(\nabla^2 + k^2)h_i = 1 + r_i$  (21)

Since this solution is axisymmetric with respect to the source, it is independent of the polar angle  $\theta$ , and thus Eq. (21) becomes [9]

$$\left(\frac{d^2}{dr_i^2} + \frac{2}{r_i} \frac{d}{dr_i} + k^2\right)h_i = 1 + r_i \quad (22)$$

A regular solution of the Eq. (22) is obtained [9-10]

$$h_i = \frac{1+r_i}{k^2} - \frac{2}{k^4} \left(\frac{1-\cos(kr_i)}{r_i}\right) \quad (23)$$

where:

$$r_i = \sqrt{(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2} \quad (24)$$

Substituting of Eqs. (18) and (19) into Eq. (15) yields

$$e(p)\phi(p) = \int_{\Gamma} \left(G \frac{\partial \phi}{\partial n} - \phi \frac{\partial G}{\partial n}\right) ds - \sum_{i=1}^{L+N} \alpha_i \int_{\Omega} G(\nabla^2 h_i + k^2 h_i) d\Omega \quad (25)$$

Using the Green's identity, for two functions of the  $G$  and  $h_i$ , integral of the last term of the Eq. (25) can be expressed in terms of the boundary integral.

$$\int_{\Omega} G(\nabla^2 h_i + k^2 h_i) d\Omega = e(p)h_i(p) - \int_{\Gamma} \left(G \frac{\partial h_i}{\partial n} - h_i \frac{\partial G}{\partial n}\right) ds \quad (26)$$

The term  $\partial h_i / \partial n$  is the normal derivative of the  $h_i$  and can be expressed as

$$h'_n = \frac{\partial h_i}{\partial n} = \frac{\partial h_i}{\partial r} \frac{\partial r}{\partial n} = \frac{\partial h_i}{\partial r} \frac{\vec{r} \cdot \vec{n}}{r} \quad (27)$$

Finally, Eq. (25) can be expressed as

$$e(p)\phi(p) = \int_{\Gamma} \left(G \frac{\partial \phi}{\partial n} - \phi \frac{\partial G}{\partial n}\right) ds - \sum_{i=1}^{N+L} \alpha_i \left\{ e(p)h_i(p) - \int_{\Gamma} \left(G h'_i - h_i \frac{\partial G}{\partial n}\right) ds \right\} \quad (28)$$

Discretization form of the Eq. (28) can be represented as follows:

$$e_l \phi_l = \sum_{j=1}^N L_{lj} \phi_{n_j} - \sum_{j=1}^N \hat{H}_{lj} \phi_j - \sum_{i=1}^{N+L} \alpha_i \left\{ e_l h_{li} - \sum_{j=1}^N L_{lj} h'_{ji} - \sum_{j=1}^N \hat{H}_{lj} h_{ji} \right\} \quad (29)$$

where  $L_{lj}$  and  $\hat{H}_{lj}$  are influence coefficients and defined as follows:

$$L_{lj} = \int_{\Gamma} G ds$$

$$\hat{H}_{lj} = \int_{\Gamma} \frac{\partial G}{\partial n} ds \quad (30)$$

These integral can be evaluated by numerical and analytical methods.

Moreover setting

$$H_{lj} = 0.5 \delta_{lj} + \hat{H}_{lj} \quad (31)$$

where  $\delta_{lj}$  is the Kronecker delta, which is defined as  $\delta_{lj} = 0$  for  $l \neq j$  and  $\delta_{lj} = 1$

for  $l = j$ , Eq. (27) may further be written as

$$\sum_{j=1}^N H_{lj} \phi_j = \sum_{j=1}^N L_{lj} \phi_{n_j} - \sum_{i=1}^{N+L} \alpha_i \left\{ \sum_{j=1}^N H_{lj} h_{ji} - \sum_{j=1}^N L_{lj} h'_{ji} \right\} \quad (32)$$

The last term of the RHS of the Eq. (32) can be calculated by the integral on the boundary. If the Neumann boundary condition employs (it means that  $\phi_{n_j}$  is known) all terms of the RHS will be completely known. Then, Eq. (32) can be arranged as

$$\sum_{j=1}^N [H]_{ij} \{\phi\}_j = \{B\}_i \quad (33)$$

Eq. (33) may be solved and the unknown variables are obtained [11].

#### IV. EVALUATION OF THE INTEGRALS

In order to evaluate the boundary integrals appearing in Eq. (30), a numerical integration strategy should be performed. To do this, the boundary  $S$  of the domain is discretized into a sufficient number of elements as it is the first stage of the boundary or finite element methods.

The regular integrals of the Eq. (30) are evaluated numerically using a standard Gaussian quadrature, but for the singular integrands, an analytical method is employed to determine the integrals.

Inserting the Green's function of Eq. (17) into the Eq. (30), we have

$$L_{lj} = \int_{\Gamma} \frac{\exp(-jkr)}{4\pi r} ds$$

$$\hat{H}_{lj} = \int_{\Gamma} \frac{\partial}{\partial n} \left( \frac{\exp(-jkr)}{4\pi r} \right) ds \quad (34)$$

where  $r$  distance between the source point and integral element i.e. expressed as

$$\vec{r} = \vec{q} - \vec{p} \quad (35)$$

Fig. 2 shows the integration element and source point.

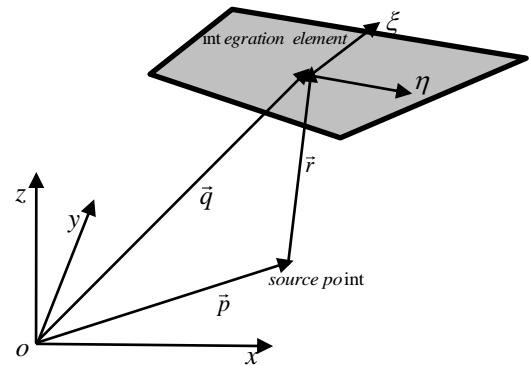


Fig. 2. Coordinate system of source point and integration element

For 3D geometry, linear or quadratic elements are adopted. In each element, the global coordinate  $\vec{q}(\xi, \eta)$  is interpolated between the coordinates  $(\vec{q}^\beta)$  of the element nodes through interpolation functions (where  $\beta$  is the node index).

$$q(\xi, \eta) = \sum_{\beta=1}^4 N_\beta(\xi, \eta) \vec{q}^\beta, \text{ linear}$$

$$q(\xi, \eta) = \sum_{\beta=1}^8 N_\beta(\xi, \eta) \vec{q}^\beta, \text{ quadratic} \quad (36)$$

$\xi$  and  $\eta$  are intrinsic coordinates both taking a value between  $-1$  to  $+1$ , and  $N_\beta(\xi, \eta)$  are the shape functions which can be expressed as

$$\begin{cases}
 N_\beta(\xi, \eta) = \frac{1}{4}(1 + \xi_\beta \xi)(1 + \eta_\beta \eta), & \beta = 1-4 \quad \text{linear} \\
 N_\beta(\xi, \eta) = \frac{1}{4}(1 + \xi_\beta \xi)(1 + \eta_\beta \eta)(-1 + \xi_\beta \xi + \eta_\beta \eta), & \beta = 1-4 \\
 N_\beta(\xi, \eta) = \frac{1}{2}(1 + \xi_\beta \xi + \eta_\beta \eta)(1 - (\xi_\beta \eta)^2 - (\eta_\beta \xi)^2), & \beta = 5-8 \quad \text{quadratic}
 \end{cases} \quad (37)$$

where  $\xi_\beta$  and  $\eta_\beta$  are nodal values of  $\xi$  and  $\eta$  (Figure 3). Two orthogonal tangential vectors are defined at point  $\vec{q}(\xi, \eta)$  of the boundary as:

$$\begin{aligned}
 \vec{a}_1 &= \frac{\partial \vec{q}(\xi, \eta)}{\partial \xi} = \sum_{\beta=1}^m \frac{\partial N_\beta(\xi, \eta) \vec{q}}{\partial \xi} \\
 \vec{a}_2 &= \frac{\partial \vec{q}(\xi, \eta)}{\partial \eta} = \sum_{\beta=1}^m \frac{\partial N_\beta(\xi, \eta) \vec{q}}{\partial \eta}
 \end{aligned} \quad (38)$$

where  $m = 4$  (linear) or  $8$  (quadratic) locally numbers the nodes within each element.

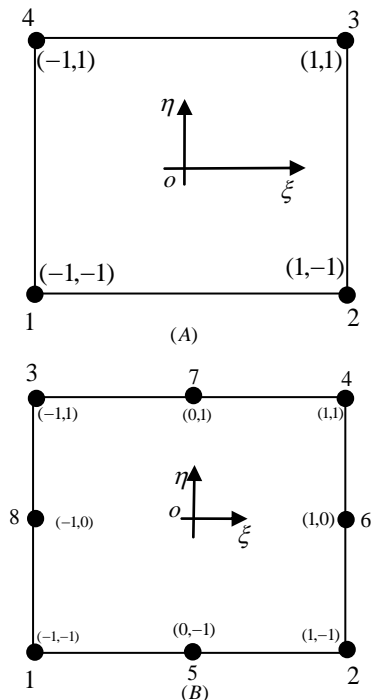


Fig. 3. Nodes over a quadrilateral element, (A) linear, (B) quadratic

After transformation from the global coordinates to the intrinsic coordinate, the Eq. (33) can be written as:

$$\begin{aligned}
 L_{ij} &= \int_{-1}^{+1} \int_{-1}^{+1} \frac{\exp(-jkr)}{4\pi r} |J(\xi, \eta)| d\xi d\eta \\
 \hat{H}_{ij} &= \int_{-1}^{+1} \int_{-1}^{+1} \frac{\partial}{\partial r} \left( \frac{\exp(-jkr)}{4\pi r} \right) \frac{\partial r}{\partial n} |J(\xi, \eta)| d\xi d\eta
 \end{aligned} \quad (39)$$

These above integrals can be determined by numerical Gaussian quadrature or analytical methods [12].

### V. ALGORITHM OF THE DRBEM

Boundary element method (BEM) is powerful numerical tool for solving many engineering problems. First step should be defined the body geometry into some collocation points or elements. In the BEM, the solutions of the Poisson or Helmholtz equations are expressed in the three integral forms in which one of them are in domain integrals. For this reason, DRM is employed to convert this domain integral into the boundary integral using RBF and internal nodal. Fig. 4 shows the flowchart algorithm of the DRBEM.

### VI. CONCLUSIONS

The Governing equation of the acoustic problem in three-dimensional potential flow was presented. The mathematical form of this physical problem is found the 3D Helmholtz equation. DRM method was applied to solve this equation. Based on this research, following conclusions can be drawn:

- The acoustic propagates inside the silencer and its sound can be formulated by the three dimensional (3D) Helmholtz equation.
- DRM is the most powerful numerical tool to solve this equation.
- The particular solution of the Helmholtz equation is found from the regular integrating.
- Evaluations of the integral can be found using Gaussian numerical and analytical method.
- Flowchart algorithm of the DRBEM was presented to calculate the unknown of the quantity.
- Special techniques should be focused to determine the analytical method of the integrals.

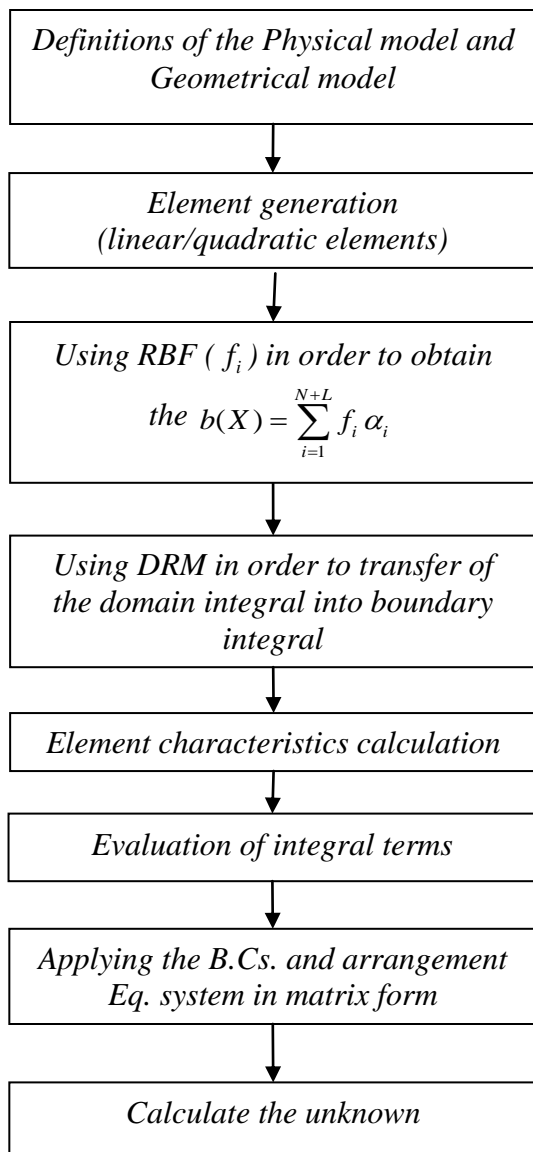


Fig. 4. Flowchart algorithm of the DRBEM

## REFERENCES

- [1]. Miao, X. H., Wang X. R, Jia, D. Qian D. J., and Pang F. Z., Substructure dual reciprocity boundary element method for prediction of acoustic attenuation performance of silencers with potential flow, APCOM & ISCM, 11-14<sup>th</sup> December, 2013, Singapore.
- [2]. Nardini, D. and Brebbia, C.A., New Approach to Vibration Analysis Using Boundary Elements, in Boundary Element Methods in Engineering, Computational Mechanics Publications 1982, Southampton and Springer-Verlag, Berlin.
- [3]. Ramachandran, P.A., Boundary Element Methods in Transport Phenomena, Computational Mechanics Publication, Southampton, Boston, 1994.
- [4]. Zakerdoost, H., Ghassemi, H., Dual reciprocity boundary element method for steady state convection-diffusion-radiation problems, International Journal of Partial Differential Equations and Applications. 2014, 2(4), pp68-71.
- [5]. Lee L, Wu T W, Zhang P. A dual-reciprocity method for acoustic radiation in a subsonic non-uniform flow, Engineering Analysis with Boundary Elements, 1994, 13, pp365-370.
- [6]. Dokumaci E., An integral equation formulation for boundary element analysis of acoustic radiation problems in viscous fluids, Journal of Sound and Vibration, 1991, 147(2), pp335-348.
- [7]. Karra C, Ben Tahar M. An integral equation formulation for boundary element analysis of propagation in viscothermal fluids. Journal of the Acoustical Society of America, 1997, 102(3), pp1311-1318.
- [8]. Zhen-Lin Ji Xue-ren Wang, Application of dual reciprocity boundary element method to predict acoustic attenuation characteristics of marine engine, exhaust silencers, Journal Marine Science Application, Vol. 7, 2008, pp102-110.
- [9]. Chen CS, Rashed YF. Evolution of thin plate spline based particular solutions for Helmholtz-type equation for the DRM. Mechanical and Research Communication 1998; 25(2), pp195-201.
- [10]. Perrey-Debaine. Analysis of convergence and accuracy of the DRBEM for axisymmetric Helmholtz-type equation, Engineering Analysis with Boundary Elements, 1999, Vol. 23, pp703-711.
- [11]. Ghassemi, H., Kohansal, A.R., Numerical evaluation of various levels of singular integrals, arising in BEM and its application in hydrofoil analysis. Applied Mathematics and Computation 213, 2009b, 277-289.
- [12]. Gao, X.W. Evaluation of regular and singular domain integrals with boundary-only discretization—theory and Fortran code, Journal of Computational and Applied Mathematics, Vol. 175, 2005, pp265-290.